维格纳函数
-- 相空间中的量子理论

王群

中国科学技术大学 / 物理学院 / 近代物理系

安徽理工大学 / 力学与光电物理学院

《极端等离子体：从夸克-胶子到聚变能》研讨会，复旦大学，上海
2025.08.12-13

Cross-talk between two fields

- 共同点：

等离子体或电浆(Plasma), 夸克胶子等离子体(Qaurk-Gluon-Plasma), 血浆(Plasma) 。整体中性，局域非中性的自由粒子体系。描写工具：流体力学+输运方程。共同的偶像：朗道(Landau damping in plasma)

- 有趣的对照：

$$E = mc^2$$

重离子碰撞：碰撞核的动能 ---》物质粒子质量 (能量变物质)

聚变反应堆：核的部分质量 ---》聚变能 (物质变能量)

Motivation

主要介绍一些早期工作和探索，引导我们找到了用**Wigner**函数(或**Wigner**分布)作为研究夸克胶子等离子体自旋极化问题的工具。

1. **Wigner**函数（或**Wigner**分布）简介
2. 探索**Wigner**函数与自旋极化之路：早期工作的一些启发
3. **Wigner**函数在夸克胶子等离子体自旋极化问题的一些应用

Single-particle distribution function in classical relativistic theory

- **Single particle distribution function in phase space $f(t,x,p)$**

$$f(t,\mathbf{x},\mathbf{p})d^3xd^3p$$

particle number in phase space volume d^3xd^3p

- **The evolution of $f(t,x,p)$ is given by the classical Boltzmann equation**

$$\frac{d}{dt}f(t,\boldsymbol{x},\boldsymbol{p})=\left(\frac{\partial}{\partial t}+\frac{\boldsymbol{p}}{E_p}\cdot\nabla_{\boldsymbol{x}}+\boldsymbol{F}\cdot\nabla_{\boldsymbol{p}}\right)f(t,\boldsymbol{x},\boldsymbol{p})=C[f]$$

$$C[f]=\int_{124}d\Gamma_{12\rightarrow p4}\left(f_1f_2-f_pf_4\right)$$

Classical feature: x and p of the particle can be determined at the same time !

Timeline for Wigner functions

- **Wigner function as quantum phase space distribtion [Wigner, 1932]**

Planck constant!

$$W(x,p)=\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dy e^{ipy/\hbar}\psi^{*}\left(x+\frac{y}{2}\right)\psi\left(x-\frac{y}{2}\right)$$

Eugene Paul Wigner
Nobel Prize 1963

- 由波函数定义、包含普朗克常数，它本质上是经典相空间分布的量子对应
- 应用领域：量子力学，量子信息，量子光学，电子器件中的量子输运，化学中的隧穿、共振和输运等

Timeline for Wigner functions

- 1946-1949, phase space representation of QM, [Groenewold (1946); Moyal (1949)]
- 1952, Bohm's particle representation with quantum potential as an alternative approach to QM in phase space [Bohm (1952)]
- Coherence and polarization of optical fields [Mandel and Wolf (1965)]
- Quantum effects in electron transport [Iafrate, Grubin, Ferry (1982); Barker and Murray (1983)]
- Elementary pattern recognition [Kumar and Carroll (1984)]
- Transport in resonant tunneling devices [Ravaioli, et al. (1985); Frensley (1987); Kluksdahl, et al. (1989)]
- Quantum optics: wave-packet spread, coherent states and squeezed states of light [Kim and Wigner (1990)]

Timeline for Wigner functions

- **Chemistry: dynamics of tunneling, resonance, and dissipation in the transport across/through a barrier, and decoherence [Tanimura and Mukamel (1994); Na and Wyatt (2003)]**
- **Wave propagation through media [Mazar (1998); Jian and Chen (1999)]**
- **Quantum dissipation, mathematical basis of the time-independent Wigner functions [Kohen, Marston and Tannor (1997); Curtright, Fairlie and Zachos (1998)]**
- **Quantum entanglement in quantum information [many authors, 2009-now]**
- **WF as a bridge connecting classical and quantum world**
- **For a recent review, see e.g., Weinbub and Ferry, Appl. Phys. Rev. 5, 041104 (2018)**

Timeline for WF in high energy nuclear physics (some early works)

- **Heinz (1983),** 使用协变**WF**推导出了非阿贝尔等离子体中的动理学方程

VOLUME 51, NUMBER 5 PHYSICAL REVIEW LETTERS 1 AUGUST 1983

Kinetic Theory for Plasmas with Non-Abelian Interactions

Ulrich Heinz
Institut für Theoretische Physik der Johann Wolfgang Goethe-Universität, D-6000 Frankfurt-am-Main 11, West Germany
(Received 21 April 1983)

- **Elze-Gyulassy-Vasak (1986),** 通过协变**WF**推导出了夸克物质的输运方程

ELSEVIER
Nuclear Physics B
Volume 276, Issues 3–4, 20 October 1986, Pages 706-728

Transport equations for the QCD quark Wigner operator

H.-Th. Elze, M. Gyulassy, D. Vasak [1]

Timeline for WF in high energy nuclear physics (some early works)

- **Vasak-Gyulassy-Elze (1987), 建立了阿贝尔等离子体的量子输运理论，第一次提出了协变WF的Clifford分解**

ANNALS OF PHYSICS **173**, 462–492 (1987)

Quantum Transport Theory for Abelian Plasmas*

DAVID VASAK, MIKLOS GYULASSY, AND HANS-THOMAS ELZE

Nuclear Science Division, Lawrence Berkeley Laboratory, University of California, Berkeley, California 94720

- **Bialynicki-Birula-Gornicki-Rafelski (1991), 提出了等时WF**

PHYSICAL REVIEW D VOLUME 44, NUMBER 6 15 SEPTEMBER 1991

Phase-space structure of the Dirac vacuum

Iwo Bialynicki-Birula,* Paweł Górnicki,* and Johann Rafelski

Department of Physics, University of Arizona, Tucson, Arizona 85721

(Received 1 April 1991)

$$W_{\alpha\beta}(\mathbf{r},\mathbf{p},t) = -\tfrac{1}{2}\int d^3s\,\exp(-i\mathbf{p}\cdot\mathbf{s}) \times \left\langle \Phi \left| \exp\left(-ie\int_{-1/2}^{1/2} d\lambda\,\mathbf{s}\cdot\mathbf{A}(\mathbf{r}+\lambda\mathbf{s},t)\right)[\Psi_\alpha(\mathbf{r}+\mathbf{s}/2,t),\Psi_\beta^\dagger(\mathbf{r}-\mathbf{s}/2,t)]\right| \Phi \right\rangle . \quad (8)$$

Timeline for WF in high energy nuclear physics (some early works)

- **Zhuang-Heinz (1996), 通过等时WF研究了QED等离子体的量子输运**

ANNALS OF PHYSICS **245**, 311–338 (1996)
ARTICLE NO. 0011

Relativistic Quantum Transport Theory for Electrodynamics*

P. ZHUANG† AND U. HEINZ

Institut für Theoretische Physik, Universität Regensburg, D-93040 Regensburg, Germany

- **自2012以来的最新发展: CME + CVE + GPE + …**

USTC, Tsinghua-U, Fudan-U, Sun-Yat-Sen-U, Academia-Sinica (Taiwan),

Shandong-U, CCNU, Goethe-U-Frankfurt, Florence-U, Jagiellonian-U, ….

- 两篇综述文章供参考

Gao-Liang-Wang, Int. J. Mod. Phys. A (2021)

Hidaka-Pu-Wang-Yang, Prog. Part. Nucl. Phys. 127 (2022) 103989

CM position and relative momentum

- **QM in dim-1: particle 1 (x_1, p_1) and particle 2 (x_2, p_2)**

$$[x_1, p_1] = i\hbar \ , \ [x_2, p_2] = i\hbar, \quad \text{any others} = 0$$

- **One can use $x = (x_1 + x_2)/2$ and $p = p_1 - p_2$ as independent phase space variables. The CM position and relative momentum are commutable**

$$[x, p] = 0$$

- **One can verify: p is conjugate to $y = x_1 - x_2$**

$$x_1 p_1 + x_2 p_2 = \left(x + \frac{y}{2}\right) p_1 + \left(x - \frac{y}{2}\right) p_2 = x(p_1 + p_2) + \frac{1}{2} y p$$

- **Wigner function $W(x, p)$ is well-defined**

Wigner functions in QM

- **Schroedinger equation**

$$i\hbar\frac{\partial}{\partial t}\Psi(t,\mathbf{x}) = H\Psi(t,\mathbf{x}), \quad H = -\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x})$$

- **Definition of Wigner function through wave function**

$$W(t,\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi\hbar)^3}\int d^3y\, e^{\frac{i\mathbf{p}\cdot\mathbf{y}}{\hbar}}\Psi^*\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right)\Psi\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right)$$

- **Properties of Wigner function: $W^*(t,x,p) = W(t,x,p)$ (real)**

$$\rho = \Psi^*(t,\mathbf{x})\Psi(t,\mathbf{x}) = \int d^3p W(t,\mathbf{x},\mathbf{p})$$

probability density

$$\mathbf{j} = \frac{i\hbar}{2m}\Big(\Psi(t,\mathbf{x})\nabla\Psi^*(t,\mathbf{x}) - \Psi^*(t,\mathbf{x})\nabla\Psi(t,\mathbf{x})\Big) = \int d^3p\frac{\mathbf{p}}{m}W(t,\mathbf{x},\mathbf{p})$$

probability current density

Kinetic Equation for Wigner functions in QM

- **Kinetic equation (no collisions or interaction) for the Wigner function**

velocity

force

$$0 = \underbrace{\frac{\partial}{\partial t}W(t,\mathbf{x},\mathbf{p}) + \frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{x}}W(t,\mathbf{x},\mathbf{p}) - [\nabla_{\mathbf{x}}V(\mathbf{x})]\cdot\nabla_{\mathbf{p}}W(t,\mathbf{x},\mathbf{p})}_{\text{Liouville or kinetic equation}}$$

$$\underbrace{-\sum_{n=1}\left(\frac{\hbar}{2}\right)^{2n}\frac{(-1)^n}{(2n+1)!}(\nabla_{\mathbf{p}}\cdot\nabla_{\mathbf{x}})^{2n+1}V(\mathbf{x})W(t,\mathbf{x},\mathbf{p})}_{\text{quantum effect at least of } O(\hbar^2)}$$

- **Wigner function satisfies the Liouville or kinetic equation at classical limit $\hbar = 0$**

Wigner functions for free scalar fields in relativistic QFT

- **Lagrangian and Euler-Lagrange equation for complex scalar fields in relativistic QFT**

$$
\begin{aligned}
\mathscr{L} &= (\partial^\mu \phi^\dagger)(\partial_\mu \phi) - m^2 \phi^\dagger \phi \\
& (\partial^2 + m^2)\phi = (\partial^2 + m^2)\phi^\dagger = 0
\end{aligned}
$$

- **Current and energy-moment tensor**

$$
\begin{aligned}
j^\mu &= i\phi^\dagger(\overrightarrow{\partial^\mu} - \overleftarrow{\partial^\mu})\phi \\
T^{\mu\nu} &= \frac{i^2}{2}\phi^\dagger(\overrightarrow{\partial^\mu} - \overleftarrow{\partial^\mu})(\overrightarrow{\partial^\nu} - \overleftarrow{\partial^\nu})\phi
\end{aligned}
$$

- **Definition of Wigner function**

thermal average

$$
W(x,p) = 2\int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \left\langle \phi^\dagger\left(x+\frac{y}{2}\right)\phi\left(x-\frac{y}{2}\right)\right\rangle
$$

Wigner functions for free scalar fields in relativistic QFT

- **EOM: mass-shell and kinetic equations**

$$\begin{aligned}
\text{quasi-on-shell} \quad & \left(p^2 - m^2 - \frac{1}{4}\hbar^2\partial_x^2\right) W(x,p) &=& \quad 0 \\
\text{Vlasov eq.} \quad & p \cdot \partial_x W(x,p) &=& \quad 0
\end{aligned}$$

- **The leading order form of WF, and current and energy-momentum tensor in leading order WF**

particle anti-particle

$$\begin{aligned}
W(x,p) &= 2\delta(p^2 - m^2)\left[\Theta(p_0) f(x,p) + \Theta(-p_0)\overline{f}(x,-p)\right] \\
j^\mu &= \int d^4p p^\mu W(x,p) = \int d^3p \frac{p^\mu}{E_p}\left[f(x,p) - \overline{f}(x,p)\right] \\
T^{\mu\nu} &= \int d^4p\, p^\mu p^\nu W(x,p) = \int d^3p \frac{p^\mu p^\nu}{E_p}\left[f(x,p) + \overline{f}(x,p)\right]
\end{aligned}$$

Single-particle distribution function in classical relativistic theory

- **Single particle distribution function in extended phase space $\boldsymbol{f(t,x,p,s)}$**

$$f(t,\boldsymbol{x},\boldsymbol{p},\boldsymbol{s})d^3\boldsymbol{x}d^3\boldsymbol{p}d^3\boldsymbol{s}$$

particle number in extended phase space volume $\boldsymbol{d\Gamma = d^3xd^3pd^3s}$

- **The evolution of $\boldsymbol{f(t,x,p,s)}$ is given by the classical Boltzmann equation**

$$\left(\frac{\partial}{\partial t}+\frac{\boldsymbol{p}}{E_p}\cdot\nabla_{\boldsymbol{x}}+\boldsymbol{F}\cdot\nabla_{\boldsymbol{p}}+\boldsymbol{F}_s\cdot\nabla_{\boldsymbol{s}}\right)f(t,\boldsymbol{x},\boldsymbol{p},\boldsymbol{s})=C[f]$$

$$C[f]=\int_{124}d\Gamma_{12\to p4}\left(f_1f_2-f_pf_4\right)$$

Classical feature: $\boldsymbol{x,p,s}$ are independent classical variables, but we know that $\boldsymbol{s}$ is entangled with $\boldsymbol{p}$ in relativistic quantum theory (Dirac theory).

Wigner functions for spin-1/2 massless (chiral) fermion in QFT: how the spin degree of freedom enters the game

Pauli (non-relativistic) fermions (can be decomposed into Pauli matrices)

$$\Psi(t,\boldsymbol{x}) = \begin{pmatrix} \Psi_\uparrow(t,\boldsymbol{x}) \\ \Psi_\downarrow(t,\boldsymbol{x}) \end{pmatrix} \Longrightarrow W(x,p) = \begin{pmatrix} W_{\uparrow\uparrow}(x,p) & W_{\uparrow\downarrow}(x,p) \\ W_{\downarrow\uparrow}(x,p) & W_{\downarrow\downarrow}(x,p) \end{pmatrix}$$

Dirac (relativistic) fermions (can be decomposed into 16 Clifford algebra generators)

$$\Psi(t,\boldsymbol{x}) = \begin{pmatrix} \Psi_1(t,\boldsymbol{x}) \\ \Psi_2(t,\boldsymbol{x}) \\ \Psi_3(t,\boldsymbol{x}) \\ \Psi_4(t,\boldsymbol{x}) \end{pmatrix} \Longrightarrow W(x,p) = \begin{pmatrix} W_{11} & W_{12} & W_{13} & W_{14} \\ W_{21} & W_{22} & W_{23} & W_{24} \\ W_{31} & W_{32} & W_{33} & W_{34} \\ W_{41} & W_{42} & W_{43} & W_{44} \end{pmatrix}(x,p)$$

Rotation and Spin in HIC

Rotation effects in HIC

- **Huge global orbital angular momenta are produced**

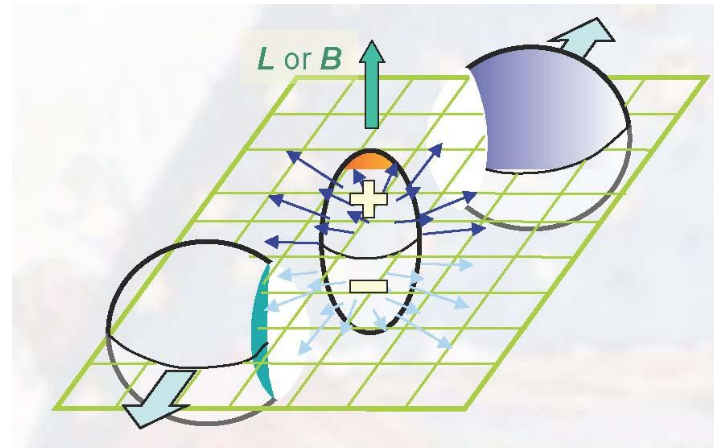
$$\mathrm{L} \sim 10^5 \hbar$$

- **Very strong magnetic fields are produced**

$$\mathrm{B} \sim m_\pi^2 \sim 10^{18}\ \mathrm{Gauss}$$

- **How do orbital angular momenta be transferred to the matter in HIC?**

- **How is spin coupled to local vorticity in the fluid?**



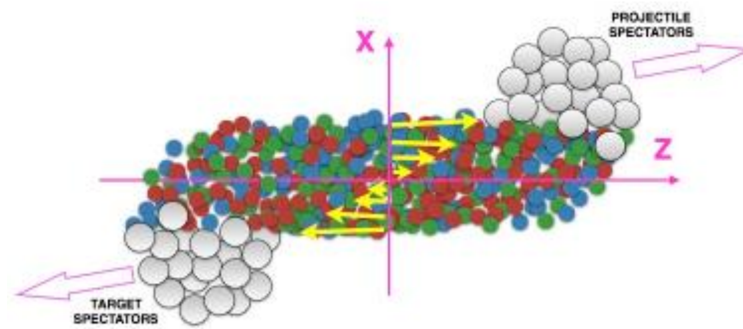


Figure taken from
Becattini et al, 1610.02506

Two classical papers on global polarization and spin alignment in HIC

Physical Review Letters

Highlights Recent Accepted Collections Authors Referees Press About Editorial Team RSS

Globally Polarized Quark-Gluon Plasma in Noncentral $A+A$ Collisions

Zuo-Tang Liang[1] and Xin-Nian Wang[2,1]

Show more

PDF

Share

Phys. Rev. Lett. **94**, 102301 – **Published 14 March, 2005** | Erratum Phys. Rev. Lett. **96**, 039901 (2006)

DOI: https://doi.org/10.1103/PhysRevLett.94.102301

Export Citation

617 citations

Two classical papers on global polarization and spin alignment in HIC

ELSEVIER

Physics Letters B

Volume 629, Issue 1, 17 November 2005, Pages 20-26

Spin alignment of vector mesons in non-central $A + A$ collisions

Zuo-Tang Liang [a], Xin-Nian Wang [a b]

Show more

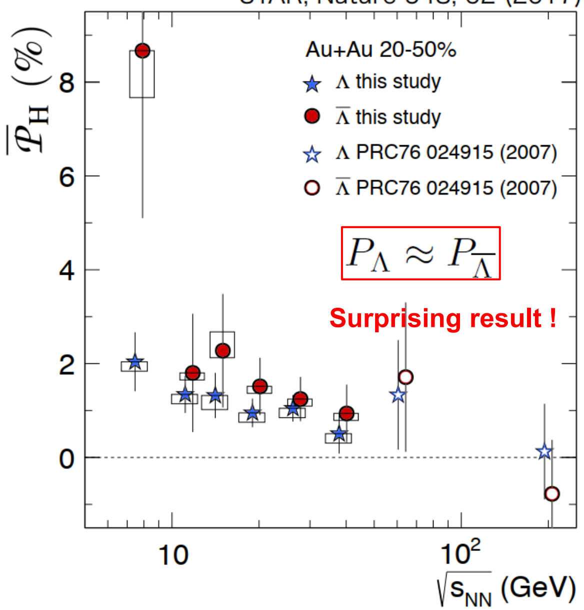
292 citations

Add to Mendeley Share Cite

https://doi.org/10.1016/j.physletb.2005.09.060

STAR: global polarization of Λ hyperon

STAR, Nature 548, 62 (2017)



parity-violating decay of hyperons

In case of Λ's decay, daughter proton preferentially decays in the direction of Λ's spin (opposite for anti-Λ)

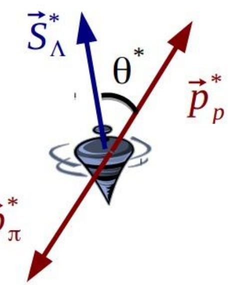
$$\frac{dN}{d\Omega^*} = \frac{1}{4\pi}(1 + \alpha \mathbf{P_\Lambda} \cdot \mathbf{p_p^*})$$

α: Λ decay parameter (=0.642±0.013)

P_Λ: Λ polarization

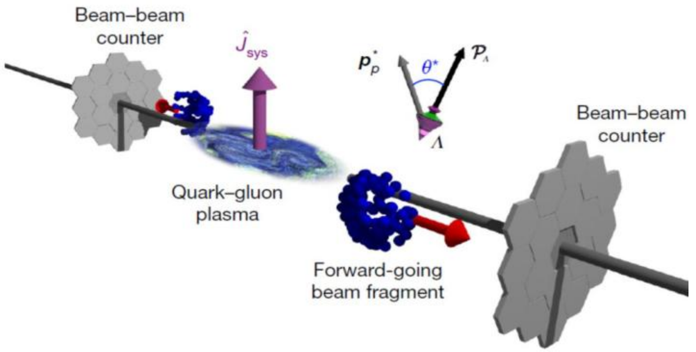
p_p^*: proton momentum in Λ rest frame

Updated by BES III, PRL129, 131801 (2022)



$\Lambda \to p + \pi^+$

(BR: 63.9%, cτ~7.9 cm)

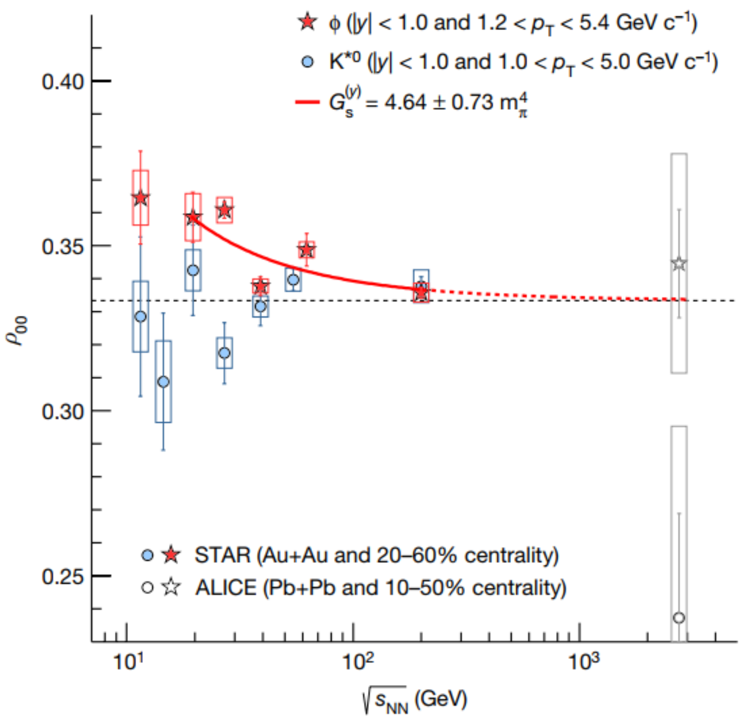


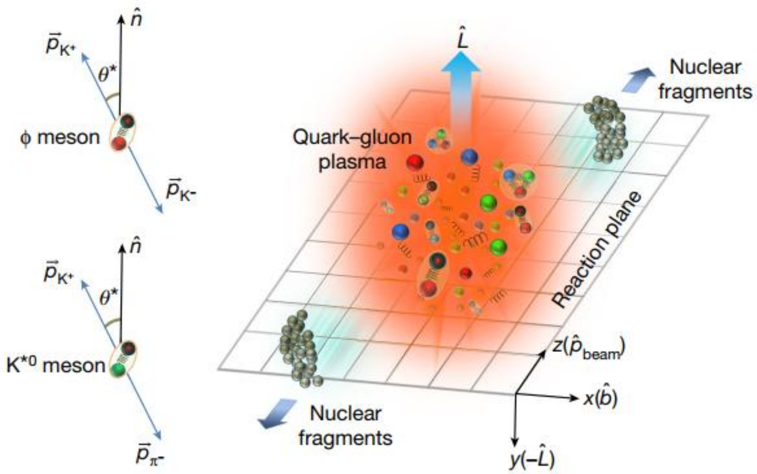
ω = (9 ± 1)x10^{21}/s, the largest angular velocity that has ever been observed in any system

Liang, Wang, PRL (2005)
Betz, Gyulassy, Torrieri, PRC (2007)
Becattini, Piccinini, Rizzo, PRC (2008)
Gao et al., PRC (2008)

STAR: global spin alignments of vector mesons

STAR, Nature 614, 244 (2023);





Implication of correlation or fluctuation in polarization s and s-bar

Theory prediction:

Liang, Wang, PLB(2005);
Sheng, Oliva, QW, PRD(2020);
Sheng, Oliva, Liang, QW, Wang, PRL(2022).

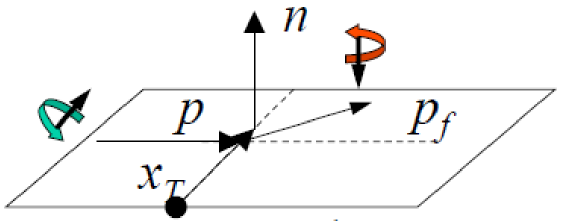
$$P_\Lambda \sim \langle P_s \rangle, \quad P_{\bar{\Lambda}} \sim \langle P_{\bar{s}} \rangle$$

$$\rho_{00}^{\phi} - \frac{1}{3} \sim \langle P_s P_{\bar{s}} \rangle \neq \langle P_s \rangle \langle P_{\bar{s}} \rangle \sim P_\Lambda P_{\bar{\Lambda}}$$

Models for global polarization through spin-orbit coupling

Quark polarization in potential scatterings

- **Quark scatterings at small angle in static potential at impact parameter x_T**
- **Unpolarized and polarized cross sections**



$$\frac{d\sigma}{d^2\vec{x}_T} = \frac{d\sigma_+}{d^2\vec{x}_T} + \frac{d\sigma_-}{d^2\vec{x}_T} = 4C_T\alpha_s^2 K_0(\mu x_T)$$

$$\frac{d\Delta\sigma}{d^2\vec{x}_T} = \frac{d\sigma_+}{d^2\vec{x}_T} - \frac{d\sigma_-}{d^2\vec{x}_T} \propto \boxed{\vec{n}\cdot(\underline{\vec{x}_T \times \vec{p}})}$$

$$A^0(q_T) = \frac{1}{q_T^2 + \mu^2}$$

screening mass

$$\boxed{\mu \sim T\sqrt{\alpha_S}}$$

Spin quantization direction **OAM** **Spin-orbit coupling**

- **Polarization for small angle scattering and** $m_q \gg p, \mu$

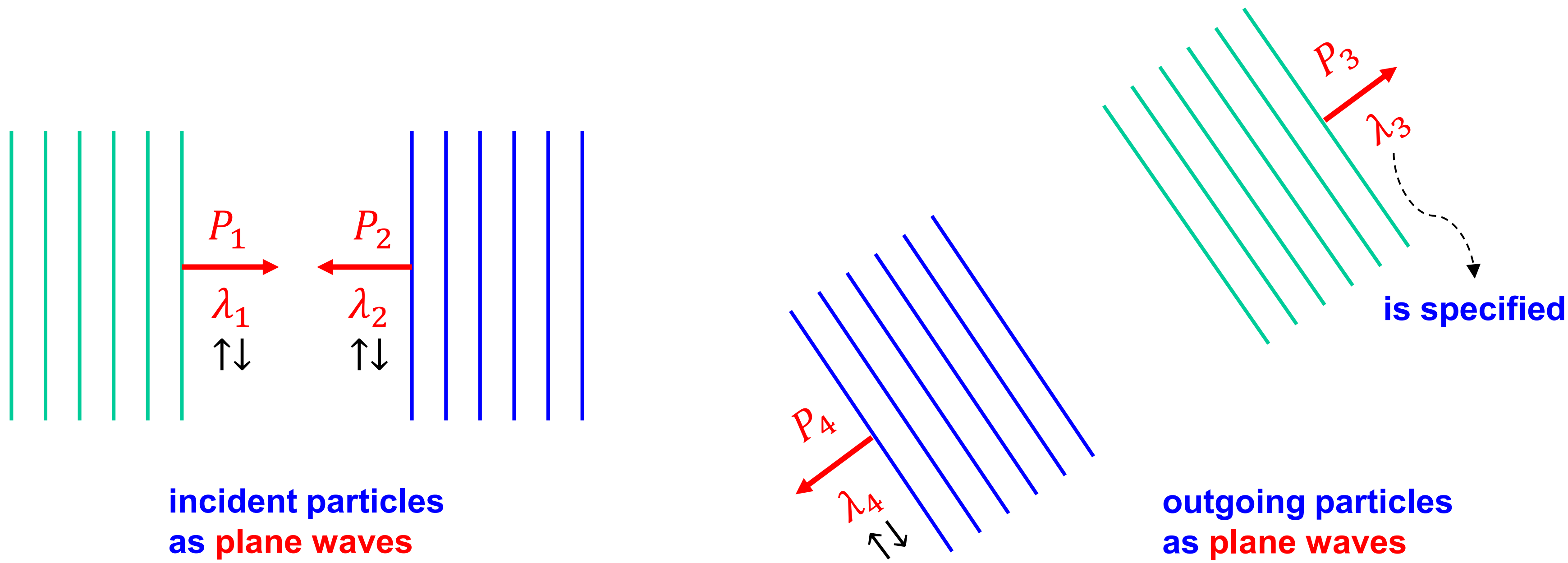
$$P_q \approx -\pi\frac{\mu p}{4m_q^2} \sim -\frac{\Delta E_{LS}}{E_0}$$

Liang, Wang, PRL 94, 102301(2005)

- **With initial polarization P_i, the final polarization P_f after one scattering is** $P_f = P_i - \frac{(1-P_i^2)\pi\mu p}{2E(E+m) - P_i\pi\mu p}.$

Huang, Huovinen, Wang, PRC84, 054910(2011)

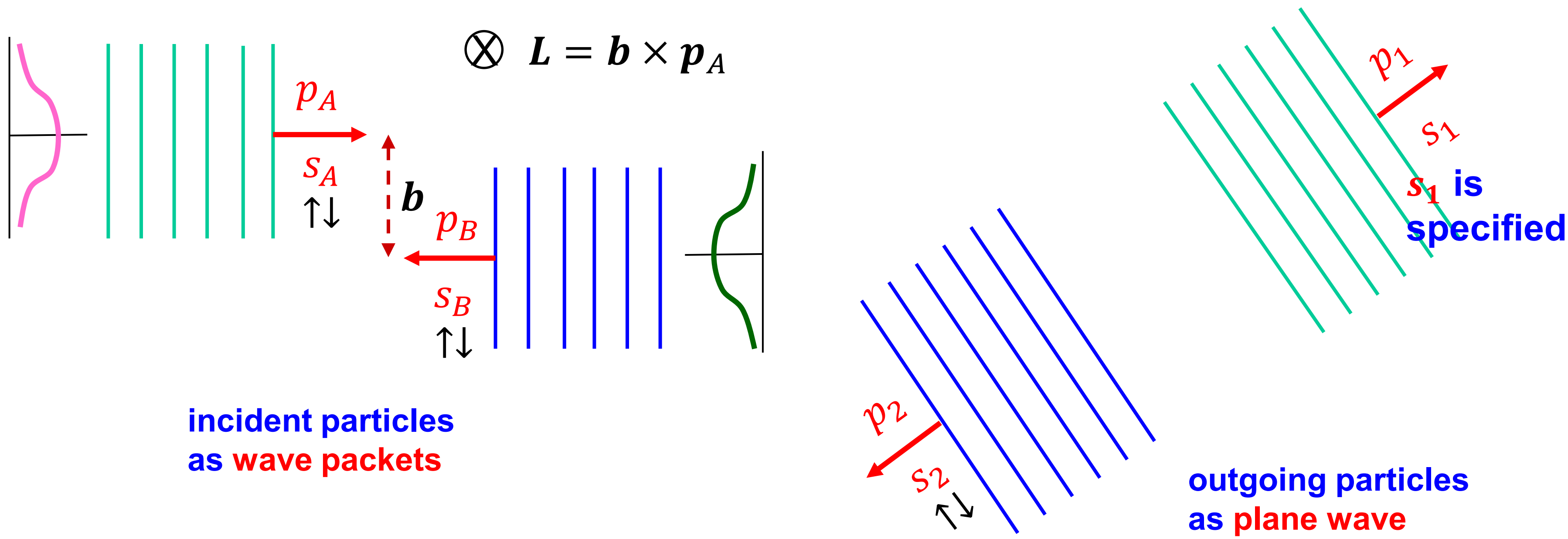
Collisions of particles as plane waves



Particle collisions as plane waves:
since there is no favored position for particles, so the OAM vanishing

$$\langle \hat{x} \times \hat{p} \rangle = 0 \quad \Longrightarrow \quad \left(\frac{d\sigma}{d\Omega}\right)_{\lambda_3=\uparrow} = \left(\frac{d\sigma}{d\Omega}\right)_{\lambda_3=\downarrow}$$

Collisions of particles as wave packets



Particle collisions as wave packets: there is a transverse distance between two wave packets (impact parameter) giving non-vanishing OAM and then the polarization of one final particle

$$\boldsymbol{L} = \boldsymbol{b} \times \boldsymbol{p}_A \quad \Longrightarrow \quad \left(\frac{d\sigma}{d\Omega}\right)_{s_1=\uparrow} \neq \left(\frac{d\sigma}{d\Omega}\right)_{s_1=\downarrow}$$

Quark-quark scattering at fixed impact parameter

For the quark-quark scattering of spin-momentum states

$$q_1(P_1,\lambda_1)+q_2(P_2,\lambda_2)\rightarrow q_1(P_3,\lambda_3)+q_2(P_4,\lambda_4)$$

where $P_i=(E_i,\vec{p}_i)$ and λ_i denote spin states, the difference cross section (λ_3 is specified)

$c_{qq}=2/9$ (color factor)

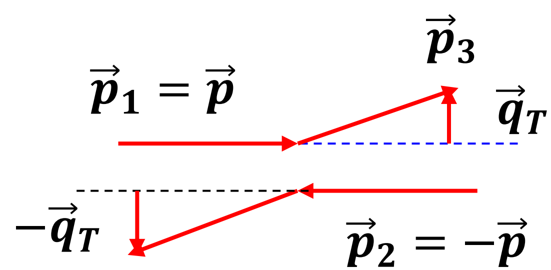
$$d\sigma_{\lambda_3}=\frac{c_{qq}}{4F}\sum_{\lambda_1\lambda_2\lambda_4}\mathcal{M}(Q)\mathcal{M}^*(Q)(2\pi)^4\delta^{(4)}(P_1+P_2-P_3-P_4)\frac{d^3\vec{p}_3}{(2\pi)^3 2E_3}\frac{d^3\vec{p}_4}{(2\pi)^3 2E_4}$$

fixed

sum over ↑↓

$Q=P_3-P_1=P_2-P_4$
(momentum transfer)

$F=4\sqrt{(P_1\cdot P_2)^2-m_1^2m_2^2}$ (flux factor)



Integrate $\vec{p}_4$ and $p_{3z}^{\pm}=\pm\sqrt{p^2-q_T^2}$

to remove $\delta^{(4)}(P_1+P_2-P_3-P_4)$

Quark-quark scattering at fixed impact parameter

We obtain $d\sigma_{\lambda_3}$ for scattered quark with spin state λ_3

$$d\sigma_{\lambda_3} = \frac{c_{qq}}{16F}\sum_{\lambda_1\lambda_2\lambda_4}\sum_{i=+,-}\frac{1}{(E_1+E_2)|p^i_{3z}|}\mathcal{M}(Q_i)\mathcal{M}^*(Q_i)\frac{d^2\vec{q}_T}{(2\pi)^2}$$

for small angle scattering, only $i=+$ is relevant

Jacobian

momentum transfer in small angle scattering

Then we can introduce impact parameter $\vec{x}_T = (x_T, \phi)$

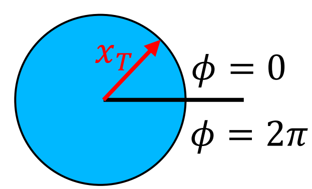
$$d\sigma_{\lambda_3} = \frac{c_{qq}}{16F}\sum_{\lambda_1,\lambda_2,\lambda_3}\int d^2\vec{x}_T\int\frac{d^2\vec{q}_T}{(2\pi)^2}\int\frac{d^2\vec{k}_T}{(2\pi)^2}\exp\left[i\left(\vec{k}_T-\vec{q}_T\right)\cdot\vec{x}_T\right]\frac{\mathcal{M}(\vec{q}_T)\mathcal{M}^*(\vec{k}_T)}{\Lambda(\vec{q}_T)\Lambda^*(\vec{k}_T)}$$

$$\| \quad \sqrt{(E_1+E_2)|p^+_{3z}(q_T)|}$$

$\Rightarrow d^2\sigma_{\lambda_3}/d^2\vec{x}_T$ $\quad \boxed{1}$

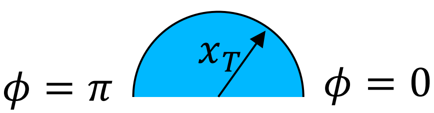
If we integrate over $\vec{x}_T$ in whole space we obtain

$$\sigma_{\lambda_3} = \int_0^\infty dx_T\, x_T \int_0^{2\pi} d\phi\, \frac{d^2\sigma_{\lambda_3}}{d^2\vec{x}_T} \Longrightarrow \sigma_\uparrow = \sigma_\downarrow$$



Quark-quark scattering at fixed impact parameter

If we integrate over $\vec{x}_T$ in half-space we obtain



$$\sigma_{\lambda_3} = \int_0^{\infty} dx_T\, x_T \int_0^{\pi} d\phi \frac{d^2\sigma_{\lambda_3}}{d^2\vec{x}_T} \Longrightarrow \sigma_{\uparrow} \neq \sigma_{\downarrow}$$

The differential cross section for spin-independent and spin-dependent part

$$\frac{d^2\sigma_{\lambda_3}}{d^2\vec{x}_T} = \frac{d^2\sigma}{d^2\vec{x}_T} + \lambda_3 \frac{d^2\Delta\sigma}{d^2\vec{x}_T}$$

$$\frac{d^2\sigma}{d^2\vec{x}_T} = \frac{1}{2}\left(\frac{d^2\sigma_{\uparrow}}{d^2\vec{x}_T} + \frac{d^2\sigma_{\downarrow}}{d^2\vec{x}_T}\right) = F(x_T)$$

$$\frac{d^2\Delta\sigma}{d^2\vec{x}_T} = \frac{1}{2}\left(\frac{d^2\sigma_{\uparrow}}{d^2\vec{x}_T} - \frac{d^2\sigma_{\downarrow}}{d^2\vec{x}_T}\right) = \underline{\vec{n}\cdot(\vec{x}_T \times \vec{p})}\Delta F(x_T)$$

spin-orbit coupling

$$\boxed{\begin{aligned}\Delta\sigma &= \int_0^{\infty} dx_T x_T \int_0^{\pi} d\phi \frac{d^2\Delta\sigma}{d^2\vec{x}_T} \\ \sigma &= \int_0^{\infty} dx_T x_T \int_0^{2\pi} d\phi \frac{d^2\Delta\sigma}{d^2\vec{x}_T}\end{aligned}}$$

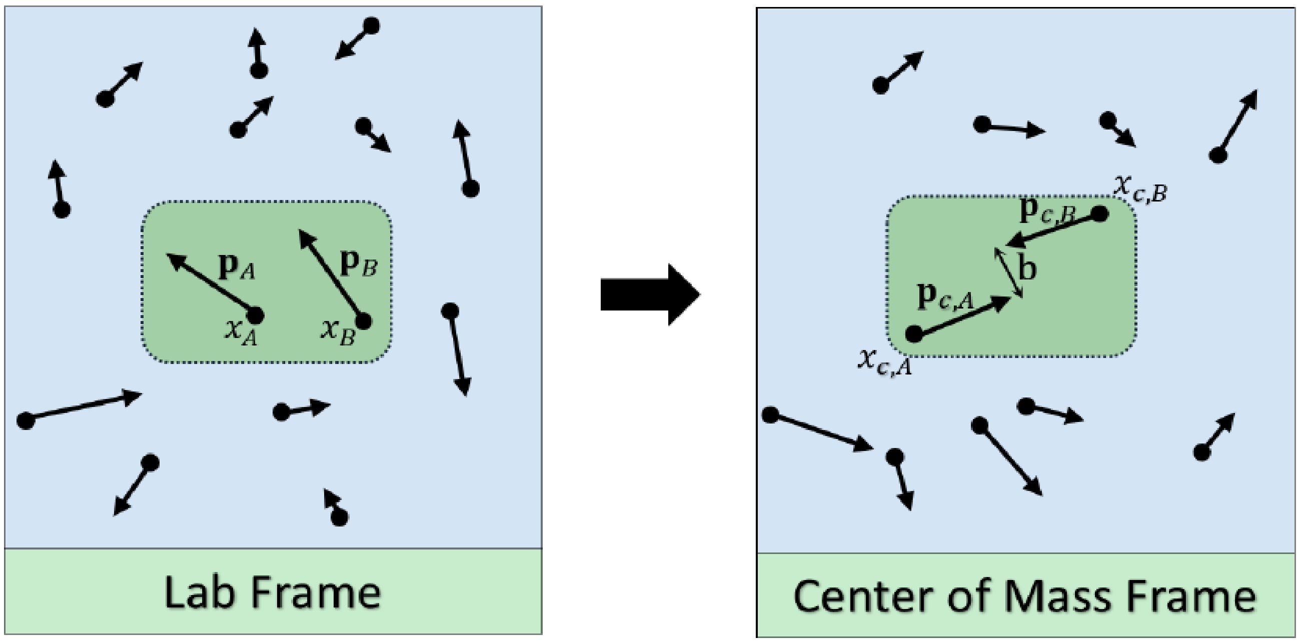
$$\Downarrow$$

$$\boxed{P_q = \frac{\Delta\sigma}{\sigma}}$$

Gao, Chen, Deng, Liang, QW, et al, PRC 77, 044902 (2008)

Emergence of spin-vorticity coupling from spin-orbit coupling

Ensemble average in thermal QGP for global polarization through spin-orbit couplings in parton scatterings



Gao, Liang, Pu, QW, Wang PRL (2012)
Zhang, Fang, QW, Wang, PRC (2019)

The first work that led us the way to the Wigner functions

PHYSICAL REVIEW D **83,** 094017 (2011)

Consistent description of kinetic equation with triangle anomaly

Shi Pu, Jian-hua Gao, and Qun Wang
Interdisciplinary Center for Theoretical Study and Department of Modern Physics,
University of Science and Technology of China, Hefei 230026, China
(Received 24 October 2010; revised manuscript received 4 March 2011; published 13 May 2011)

We provide a consistent description of the kinetic equation with a triangle anomaly which is compatible with the entropy principle of the second law of thermodynamics and the charge/energy-momentum conservation equations. In general an anomalous source term is necessary to ensure that the equations for the charge and energy-momentum conservation are satisfied and that the correction terms of distribution functions are compatible to these equations. The constraining equations from the entropy principle are derived for the anomaly-induced leading order corrections to the particle distribution functions. The correction terms can be determined for the minimum number of unknown coefficients in one charge and two charge cases by solving the constraining equations.

DOI: 10.1103/PhysRevD.83.094017 PACS numbers: 12.38.Mh, 25.75.Nq

Our work is inspired by Son, Surowka, PRL (2009) [CVE and CME unified in hydrodynamics based on entropy principle]

The first work that led us the way to the Wigner functions

II. CONSTRAINING DISTRIBUTION FUNCTION WITH ANOMALY COMPATIBLE TO SECOND LAW OF THERMODYNAMICS

In this and the next sections we will consider the most simple case with one charge and one particle species (without antiparticles). The relativistic Boltzmann equation for the on shell phase space distribution $f(x, p)$ in a background electromagnetic field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is given by

$$p^\mu\left(\frac{\partial}{\partial x^\mu} - QF_{\mu\nu}\frac{\partial}{\partial p_\nu}\right)f(x, p) = \mathcal{C}[f], \qquad (1)$$

where the charge of the particle is $Q = \pm 1$. Here p denotes the on shell 4-momentum satisfying $p^2 = m^2$ where m is the particle mass. We note that $\mathcal{C}[f]$ contains a normal collision term $\mathcal{C}_0[f]$ and a source term from anomaly $\mathcal{C}_A[f]$, $\mathcal{C}[f] = \mathcal{C}_0[f] + \mathcal{C}_A[f]$. We assume that $\mathcal{C}_A[f]$ is at most of the first order, a small quantity. The necessity for the source term is to make the charge conservation equation hold:

$$\partial_\mu j^\mu = -CE^\mu B_\mu \equiv -CE \cdot B. \qquad (2)$$

Here j^μ is the charge current and $E^\mu = u_\nu F^{\mu\nu}$ and $B_\mu = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}u^\nu F^{\alpha\beta}$ are electric and magnetic field vectors, respectively, where u_μ is the fluid velocity and $\epsilon_{\mu\nu\alpha\beta} = -\epsilon^{\mu\nu\alpha\beta} = -1, 1$ for the order of Lorentz indices $(\mu\nu\alpha\beta)$ is an even/odd permutation of (0123). However, the presence of the source term should not influence the energy-momentum conservation:

$$\partial_\mu T^{\mu\nu} = F^{\nu\mu} j_\mu. \qquad (3)$$

J.H. Gao, S. Pu, QW, Phys. Rev. D83, 094017 (2011)

The first work that led us the way to the Wigner functions

We assume that the distribution function f in presence of an anomaly is a solution of the Boltzmann equation with collision terms in Eq. (1), where β, u_μ and μ are functions of space-time. Generally $f(x,p)$ can be written in the following form:

$$f(x,p) = \frac{1}{e^{(u\cdot p - Q\mu)/T + \chi(x,p)} - e} = f_0(x,p) + f_1(x,p), \tag{6}$$

where $f_0(x,p)$ is given in Eq. (5) and $f_1(x,p)$ is the first order deviation from it:

$$f_1(x,p) = -f_0(x,p)[1 + e f_0(x,p)]\chi(x,p). \tag{7}$$

It is known that a magnetic field is closely related to a charge rotation characterized by vorticity. So we introduce into the distribution function terms associated with the vorticity-induced current $\omega_\mu = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}u^\nu\partial^\alpha u^\beta$ and the magnetic field 4-vector B_μ which are assumed to be of the first order, which provide a leading order correction to the particle distribution function. For simplicity we will neglect viscous and diffusive effects throughout the paper, then the ordinary form in the current scheme for $\chi(x,p)$ reads

$$\chi(x,p) = \lambda(p) p\cdot\omega + \lambda_B(p) p\cdot B, \tag{8}$$

where $\lambda(p)$ and $\lambda_B(p)$ are functions of μ, T, and $u\cdot p$ and have mass dimension -2 and -3, respectively. We will show that $\lambda(p)$ and $\lambda_B(p)$ must depend on momentum otherwise they will contradict the entropy principle from the second law of thermodynamics.

J.H. Gao, S. Pu, QW,
Phys. Rev. D83, 094017 (2011)

The first work that led us the way to the Wigner functions

Using Eq. (6) we can decompose the charge and entropy currents and the stress tensor into equilibrium values and the leading order (first order) corrections as $j^\mu = j_0^\mu + j_1^\mu$, $S^\mu = S_0^\mu + S_1^\mu$ and $T^{\mu\nu} = T_0^{\mu\nu} + T_1^{\mu\nu}$ with

$$\begin{aligned} j^\mu_{0,1}(x) &= Q\int[dp]p^\mu f_{0,1}(x,p), \\ S_0^\mu(x) &= -\int[dp]p^\mu \psi(f_0), \\ S_1^\mu(x) &= -\int[dp]p^\mu \psi'(f_0)f_1, \\ T^{\mu\nu}_{0,1}(x) &= \int[dp]p^\mu p^\nu f_{0,1}(x,p), \end{aligned} \tag{9}$$

where we have defined $[dp] \equiv d_g \frac{d^3p}{(2\pi)^3(u\cdot p)}$ (d_g is the degeneracy factor), $\psi(f_0) = f_0 \ln(f_0) - e(1+ef_0)\times \ln(1+ef_0)$ and $\psi'(f_0) = \ln[f_0/(1+ef_0)] = -(u\cdot p - Q\mu)/T$. Inserting f_0 into the above formula, we obtain the charge and entropy currents and the stress tensor in equilibrium, $j_0^\mu = nu^\mu$, $S_0^\mu = su^\mu$ and $T_0^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - Pg^{\mu\nu}$, with the energy density ε, the pressure P, the particle number density n and the entropy density $s = (\epsilon + P - n\mu)/T$. Using Eqs. (7)–(9), we obtain

$$\begin{aligned} j_1^\mu &= \xi\omega^\mu + \xi_B B^\mu, \\ T_1^{\mu\nu} &= DT(u^\mu\omega^\nu + u^\nu\omega^\mu) + D_B T(u^\mu B^\nu + u^\nu B^\mu), \\ S_1^\mu &= -\frac{\mu}{T}(\xi\omega^\mu + \xi_B B^\mu) + (D\omega^\mu + D_B B^\mu), \end{aligned} \tag{10}$$

where

$$\begin{aligned} \xi &= -QJ_{21}^{\lambda} \equiv \frac{1}{3}Q\int[dp][(p\cdot u)^2 - m^2]f_0(1+ef_0)\lambda(p), \\ \xi_B &= -QJ_{21}^{\lambda_B} \equiv \frac{1}{3}Q\int[dp][(p\cdot u)^2 - m^2]f_0(1+ef_0)\lambda_B(p), \\ D &= -\frac{J_{31}^{\lambda}}{T} \\ &\equiv \frac{1}{3T}\int[dp][(p\cdot u)^2 - m^2](p\cdot u)f_0(1+ef_0)\lambda(p), \\ D_B &= -\frac{J_{31}^{\lambda_B}}{T} \\ &\equiv \frac{1}{3T}\int[dp][(p\cdot u)^2 - m^2](p\cdot u)f_0(1+ef_0)\lambda_B(p). \end{aligned} \tag{11}$$

J.H. Gao, S. Pu, QW,
Phys. Rev. D83, 094017 (2011)

The first work that led us the way to the Wigner functions

On the other hand, ξ, ξ_B, D and D_B as functions of μ and T can be determined by the second law of thermodynamics or the entropy principle together with Eqs. (2) and (3). We find that $\partial_\mu(su^\mu + S_1^\mu)$ cannot be positive definite unless we make a shift to introduce a new entropy current $\tilde{S}^\mu$ as follows:

$$\begin{aligned}\tilde{S}^\mu &= su^\mu + S_1^\mu - (D\omega^\mu + D_B B^\mu)\\ &= su^\mu - \frac{\mu}{T}(\xi\omega^\mu + \xi_B B^\mu)\\ &= \frac{1}{T}(Pu^\mu - \mu j^\mu + u_\lambda T^{\lambda\mu}) - (D\omega^\mu + D_B B^\mu). \quad (12)\end{aligned}$$

We will use the thermodynamic relation

$$\partial_\mu(Pu^\mu) = j_0^\mu \partial_\mu \bar{\mu} - T_0^{\lambda\mu}\partial_\mu u_\lambda \qquad (13)$$

and the identities

$$\begin{aligned}u^\mu u^\lambda \partial_\mu \omega_\lambda &= \frac{1}{2}\partial_\mu \omega^\mu,\\ u^\mu u^\lambda \partial_\mu B_\lambda &= \partial_\mu B^\mu - 2\omega^\rho E_\rho,\\ \partial_\mu \omega^\mu &= -\frac{2}{\epsilon + P}(n\omega^\mu E_\mu + \omega^\mu \partial_\mu P),\\ \partial_\mu B^\mu &= 2\omega^\rho E_\rho - \frac{1}{\epsilon + P}(nB_\lambda E^\lambda + B^\mu \partial_\mu P),\end{aligned} \qquad (14)$$

to evaluate $\partial_\mu \tilde{S}^\mu$. We have used the shorthand notation $\bar{\mu} \equiv \mu/T$ in Eq. (13). Following the same procedure as in Ref. [20], we obtain

$$\begin{aligned}\partial_\mu \tilde{S}^\mu = {}& \omega^\mu\left[\xi^{SS}\partial_\mu\bar{\mu} - \partial_\mu D + \frac{2D}{\epsilon + P}\partial_\mu P\right]\\ &+ B^\mu\left[\xi_B^{SS}\partial_\mu\bar{\mu} - \partial_\mu D_B + \frac{D_B}{\epsilon + P}\partial_\mu P\right]\\ &+ E\cdot\omega\left[\frac{1}{T}\xi^{SS} + \frac{2nD}{\epsilon + P} - 2D_B\right]\\ &+ E\cdot B\left[\frac{1}{T}\xi_B^{SS} + C\frac{\mu_A}{T} + \frac{nD_B}{\epsilon + P}\right],\end{aligned} \qquad (15)$$

where we have defined

$$\xi^{SS} = \frac{DTn}{\epsilon + P} - \xi, \qquad \xi_B^{SS} = \frac{D_B T n}{\epsilon + P} - \xi_B. \qquad (16)$$

J.H. Gao, S. Pu, QW,
Phys. Rev. D83, 094017 (2011)

The first work that led us the way to the Wigner functions

For the constraint $\partial_\mu \tilde{S}^\mu \geq 0$ to hold, we impose that all quantities inside the square brackets should vanish. We finally obtain

$$D = \frac{1}{3} C \frac{\mu^3}{T}, \qquad D_B = \frac{1}{2} C \frac{\mu^2}{T},$$
$$\xi = -C \frac{sT\mu^2}{\epsilon + P}, \qquad \xi_B = -C \frac{sT\mu}{\epsilon + P}. \tag{17}$$

Using Eqs. (16) and (17), one can verify that the values of ξ^{SS} and ξ_B^{SS} are identical to Ref. [20]. The difference between our values in Eq. (17) and those in Ref. [20] arises from the fact that we do not use the Landau frame, while the authors of Ref. [20] do. By equating Eq. (11) and (17), we obtain equations for λ and λ_B

$$QJ_{21}^{\lambda} = -\xi, \qquad J_{31}^{\lambda} = -DT,$$
$$QJ_{21}^{\lambda_B} = -\xi_B, \qquad J_{31}^{\lambda_B} = -D_B T. \tag{18}$$

Equation (18) forms a complete set of constraints for λ and λ_B. We note that λ and λ_B must depend on momentum in general. If λ and λ_B are constants, we would obtain

$$\frac{\xi}{DT} = \frac{\xi_B}{D_B T} = Q \frac{J_{21}}{J_{31}}, \tag{19}$$

which contradicts Eq. (17) from the entropy principle.

We can expand $\lambda(p)$ and $\lambda_B(p)$ in powers of $u \cdot p$:

$$\lambda(p) = \sum_{i=0} \lambda_i (u \cdot p)^i, \qquad \lambda_B(p) = \sum_{i=0} \lambda_i^B (u \cdot p)^i. \tag{20}$$

So we obtain the following expressions:

$$J_{n1}^{\lambda} = \sum_{i=0} \lambda_i J_{i+n,1}, \qquad J_{n1}^{\lambda_B} = \sum_{i=0} \lambda_i^B J_{i+n,1}, \tag{21}$$

for $n = 2, 3$. Here the functions J_{nq} are integrals defined in Ref. [7,11]:

$$J_{nq} = (-1)^q \frac{1}{(2q+1)!!} \int \frac{d^3 p}{(2\pi)^3 (u \cdot p)} [(u \cdot p)^2 - m^2]^q \times (u \cdot p)^{n-2q} f_0 (1 + e f_0). \tag{22}$$

Using Eqs. (20) and (21) in Eq. (18), we can constrain the coefficients λ_i and λ_i^B. If we expand both $\lambda(p)$ and $\lambda_B(p)$ to the first power of $u \cdot p$, we can completely fix the coefficients $\lambda_{0,1}$ and $\lambda_{0,1}^B$ from Eq. (18) since we have two equations for $\lambda_{0,1}$ and two for $\lambda_{0,1}^B$:

$$\begin{pmatrix} QJ_{21} & QJ_{31} \\ J_{31} & J_{41} \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} -\xi \\ -DT \end{pmatrix}. \tag{23}$$

J.H. Gao, S. Pu, QW,
Phys. Rev. D83, 094017 (2011)

Comments: the way to the Wigner functions

1. 这个工作使用了相空间分布函数f(x,p)和微扰展开方法，能描写CVE, CME和反常项，也能与流体力学的热力学第二定律融合，但是因为使用了经典分布函数，我们总是觉得没有彻底解决问题(所谓"得道").
2. 经过一段时间思考，我们寻找到文献[Vasak, Gyulassy, Elze, Ann. Phys. 173, 462 (1987)]，这篇文章解出在背景电磁场下有质量费米子体系(QED plasma)的协变Wigner函数的解析表达式而且第一次引进了Wigner函数的Clifford分解。我们发现如果把费米子质量设为零，即手征费米子， Wigner函数满足的方程就会大大简化，纠缠的16分量的方程解耦合为对左手和右手费米子的两个独立方程组，形式完全一样(只差螺旋度符号)。我们把它应用于夸克物质，发现其解析解涵盖了CVE (LPE), CME和反常效应. 这是第一次揭示出手征费米子协变Wigner函数包含有CVE(LPE), CME和反常效应，为后续工作开辟了一条道路. [Gao, Liang, Pu, QW, Wang, PRL (2012)]

Wigner functions for massless fermions

Physical Review Letters

Highlights Recent Accepted Collections Authors Referees Press About Editorial Team RSS

Chiral Anomaly and Local Polarization Effect from the Quantum Kinetic Approach

PDF

Jian-Hua Gao[1,2], Zuo-Tang Liang[3], Shi Pu[2], Qun Wang[2], and Xin-Nian Wang[4,5]

Share

Show more

Phys. Rev. Lett. **109**, 232301 – **Published 4 December, 2012**

DOI: https://doi.org/10.1103/PhysRevLett.109.232301

266 citations

Export Citation

Based on J.H. Gao, S. Pu, QW, Phys. Rev. D83, 094017 (2011); Inspired by Vasak, Gyulassy, Elze, Ann. Phys. 173, 462 (1987)

Some later developments in Wigner functions

1. **Berry Curvature and Four-Dimensional Monopoles in the Relativistic Chiral Kinetic Equation, Chen, Pu, QW, Wang, PRL 110, 262301(2013).** 协变手征动理学方程
2. **Disentangling covariant Wigner functions for chiral fermions, Gao, Liang, QW, Wang, PRD 98, 036019(2018).** 手征维格纳函数解耦定理(到$\hbar$的任意阶)
3. **Chiral vortical effect in Wigner function approach, Gao, Pang, QW, PRD 100, 016008 (2019). CVE**分解的"**2/3**"迷失疑难
4. **Second-order charge currents and stress tensor in a chiral system, Yang, Gao, Liang, QW, PRD 102, 116024 (2020).** 电磁场中维格纳函数完整的二阶理论
5. **Polarization of massive fermions in a vortical fluid, Fang, Pang, QW, Wang, PRC 94, 024904 (2016).** 有质量费米子维格纳函数部分解析解
6. **A microscopic description for polarization in particle scatterings, Zhang, Fang, QW, Wang, PRC 100, 064904 (2019).** 利用系综平均从自旋轨道耦合推导出自旋涡旋耦合

Some later developments in Wigner functions

7. **Kinetic theory for massive spin-1/2 particles from the Wigner-function formalism, Weickgenannt, Sheng, Speranza, QW, Rischke, PRD 100, 056018 (2019); Relativistic Quantum Kinetic Theory for Massive Fermions and Spin Effects, Gao, Liang, PRD 100, 056021 (2019).** 有质量费米子的维格纳函数全解析解
8. **From Kadanoff-Baym to Boltzmann equations for massive spin-1/2 fermions, Sheng, Weickgenannt, Speranza, Rischke, QW, PRD 104, 016029 (2021).** 包含非局域碰撞项的自旋动理学方程
9. **Generating Spin Polarization from Vorticity through Nonlocal Collisions, Weickgenannt, Speranza, Sheng, QW, Rischke, PRL 127, 052301 (2021); PRD 104, 016022 (2021)**；从**WF**推导出含有非局域碰撞项的玻尔兹曼方程
10. **Spin Alignment of Vector Mesons in Heavy-Ion Collisions, Sheng, Oliva, Liang, QW, Wang, PRL 131, 042304(2023); PRD 109, 036004 (2024).** 矢量介子的动理学方程

Later developments in Wigner functions by other groups

除了中科大和山大组，很多其他研究组在夸克胶子等离子体维格纳函数方面也做出了重要贡献 (姓氏拼音排序):

Becattini (意大利弗洛伦萨大学)，**Florkowski** (波兰Jagiellonian大学)，**Hidaka**(日本京都大学), 侯德富(华中师大)，黄旭光(复旦)， 林树(中山)，**Rischke** (德国法兰克福大学), 杨迪伦 (台湾中研院)，庄鹏飞(清华)

其他研究组 ....

结束语

协变维格纳函数是描述夸克胶子等离子体自旋极化问题的有力工具，最初是中国学者做出了原创成果，经中外许多研究组共同开拓，现在已经成为广泛的前沿研究领域。

Back-up slides

Wigner functions for massless fermions

- **The Lagrangian for massless fermions in a background electromagnetic (EM) field**

χ_R and χ_L : Pauli spinors

$$\mathcal{L} = \overline{\psi} i\gamma\cdot D\psi = \chi_R^\dagger i\sigma\cdot D\chi_R + \chi_L^\dagger i\overline{\sigma}\cdot D\chi_L$$

$D_\mu = \hbar\partial_\mu + iA_\mu$ $\quad$ $\psi = (\chi_L, \chi_R)^T$ $\quad$ $\sigma^\mu = (1, \boldsymbol{\sigma}),\ \ \overline{\sigma}^\mu = (1, -\boldsymbol{\sigma})$

- **Wigner function (2x2 matrix) for massless fermions in background EM fields**

For each chirality, a 2x2 Hermitian matrix $W = W^+$ has 4 real variables

$$W_{ab}(x,p) = \int \frac{d^4y}{(2\pi)^4} \exp\left(\frac{i}{\hbar}p\cdot y\right) \left\langle \chi_b^\dagger(x_2)\chi_a(x_1)\right\rangle \underline{U(x_2,x_1)}$$

Gauge link as phase factor

$x = \frac{1}{2}(x_1+x_2)$ $\quad$ $y = x_1 - x_2$

essemble average of two-point Green function

Equation for Wigner function of massless fermions

- **EOM for Wigner functions**

Vasak-Gyulassy-Elze, Ann. Phys. 173, 462 (1987)
Gao-Liang-Pu-QW-Wang, PRL (2012)

$$\sigma \cdot \left(\frac{1}{2} i\hbar \nabla + \Pi \right) W(x,p) = 0$$

- **where the operators Δ and Π are defined as**

$$\begin{aligned} \nabla_\mu &= \partial_\mu^x - j_0(\Delta) F_{\mu\nu}(x) \partial_p^\nu \\ \Pi_\mu &= p_\mu - \frac{1}{2}\hbar j_1(\Delta) F_{\mu\nu}(x) \partial_p^\nu \end{aligned}$$

$$j_0(z) = \frac{\sin z}{z}$$

spherical Bessel function

$$\Delta = \frac{1}{2}\hbar \partial_x \cdot \partial_p$$ **with** ∂_x **acts only on** $F_{\mu\nu}(x)$

$$j_1(z) = \frac{1}{z^2}(\sin z - z\cos z)$$

spherical Bessel function

- **Leading order** $O(\hbar^0)$

$$\boxed{\begin{aligned} \nabla_\mu &= \partial_\mu^x - F_{\mu\nu} \partial_p^\nu \\ \Pi_\mu &= p_\mu \end{aligned}}$$

Wigner functions and kinetic equation for massless fermions

- **Wigner function has left- and right-handed part**

$$W_R(x,p) = \overline{\sigma}^{\mu}\,\mathscr{J}_{\mu}^{+}, \quad W_L(x,p) = \sigma^{\mu}\,\mathscr{J}_{\mu}^{-}$$

$$\mathscr{J}_{\mu}^{+} = \frac{1}{2}\mathrm{Tr}\,(\sigma_{\mu} W_R), \quad \mathscr{J}_{\mu}^{-} = \frac{1}{2}\mathrm{Tr}\,(\overline{\sigma}_{\mu} W_L)$$

$$\mathscr{J}_{\mu}^{\pm} = \underbrace{\mathscr{V}_{\mu}}_{\text{vector component}} \pm \underbrace{\mathscr{A}_{\mu}}_{\text{axial vector component}}$$

- **Master equations for right-handed and left-handed components of Wigner functions**

Vasak-Gyulassy-Elze (1987); Gao-Liang-Pu-QW-Wang (2012); ……

$$\Pi^{\mu}\,\mathscr{J}_{\mu}^{s}(x,p) = 0 \quad \Longrightarrow \quad \text{1x2 equation}$$

$$\nabla^{\mu}\,\mathscr{J}_{\mu}^{s}(x,p) = 0 \quad \Longrightarrow \quad \text{1x2 equation}$$

$$\underline{2s\,(\Pi^{\mu}\,\mathscr{J}_{s}^{\nu} - \Pi^{\nu}\,\mathscr{J}_{s}^{\mu}) = -\hbar\epsilon^{\mu\nu\rho\sigma}\nabla_{\rho}\mathscr{J}_{\sigma}^{s}} \quad \Longrightarrow \quad \text{6x2=12 equations}$$

Connecting higher to lower order Wigner functions

Semi-classical expansion and solution to Wigner functions

- **Leading (zeroth) order solution to the Wigner function at $O(\hbar^0)$**

$$\mathscr{J}^{\rho}_{(0)}(x,p) = p^{\rho} f_{(0)}(x,p)\delta(p^2)$$

- **where the distribution function for chiral fermions**

$$f_{(0)}(x,p) = \frac{2}{(2\pi)^3}\left\{\Theta(p_0) f_{\mathrm{FD}}(p_0 - \mu_s) + \Theta(-p_0)\left[f_{\mathrm{FD}}(-p_0+\mu_s) - 1\right]\right\}$$

$p_0 \equiv u \cdot p$

$\mu_s = \mu + s\mu_5$

$f_{\mathrm{FD}}(y) \equiv \dfrac{1}{\exp(\beta y) + 1}$

Fermi-Dirac distribution

- **The zeroth order solution must satisfy**

$$\begin{aligned} 0 &= \nabla_\rho \mathscr{J}^{\rho}_{(0)} \\ &= \delta(p^2) p^{\rho} \nabla_\rho f_{(0)} \end{aligned}
\quad\Longrightarrow\quad
\begin{aligned} \partial_\rho \beta_\sigma + \partial_\sigma \beta_\rho &= 0 \\ \partial_\rho \overline{\mu} + F_{\rho\sigma}\beta^{\sigma} &= 0 \\ \partial_\rho \overline{\mu}_5 &= 0 \end{aligned}$$

Killing equation

global equilibrium condition

First and second order solution to Wigner functions for massless fermions

- **Next-to-leading (first) order solution to Wigner function at $O(\hbar^1)$**

$$\mathscr{J}^{\mu}_{(1)} = -\frac{s}{2}\widetilde{\Omega}^{\mu\lambda}p_\lambda f'_{(0)}\delta(p^2) + s\widetilde{F}^{\mu\nu}p_\nu f_{(0)}\delta'(p^2)$$

Gao-Liang-Pu -QW-Wang (2012)

$$\widetilde{\Omega}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\Omega_{\rho\sigma} \quad \Omega_{\mu\nu} = \frac{1}{2}(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu) \quad \widetilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$$

- **Next-to-next-to-leading (second) order solution to Wigner function at $O(\hbar^2)$**

Yang-Gao-Liang-QW (2020)

$$\begin{aligned}\mathscr{J}^{(2)}_{\mu} &= X^{(2)}_{\mu}\delta(p^2) + \frac{s}{2p^2}\epsilon_{\mu\nu\rho\sigma}p^\nu\nabla^\rho\mathscr{J}^{\sigma}_{(1)} \\ &= \frac{1}{4p^2}\left(p_\mu\Omega_{\gamma\beta}p^\beta - p^2\Omega_{\gamma\mu}\right)\Omega^{\gamma\lambda}p_\lambda f''_{(0)}\delta(p^2) \longrightarrow \text{vorticity-vorticity} \\ &\quad + \frac{1}{p^4}\left(p_\mu F_{\gamma\beta}p^\beta - p^2F_{\gamma\mu}\right)\Omega^{\gamma\lambda}p_\lambda f'_{(0)}\delta(p^2) \longrightarrow \text{vorticity-field} \\ &\quad + \frac{2}{p^6}\left(p_\mu F_{\gamma\beta}p^\beta - p^2F_{\gamma\mu}\right)F^{\gamma\lambda}p_\lambda f_{(0)}\delta(p^2) \longrightarrow \text{field-field}\end{aligned}$$

Charge and axial charge currents up to $O(\hbar^2)$

- **The currents can be obtained by integrating out four-momenta**

$$j_s^\mu = \int d^4p\, \mathscr{J}_s^\mu(x,p) \longrightarrow \begin{aligned} j^\mu &= j_+^\mu + j_-^\mu = j_{(0)}^\mu + \hbar j_{(1)}^\mu + \hbar^2 j_{(2)}^\mu \\ j_5^\mu &= j_+^\mu - j_-^\mu = j_{5,(0)}^\mu + \hbar j_{5,(1)}^\mu + \hbar^2 j_{5,(2)}^\mu \end{aligned}$$

- **The charge (vector) currents are given by**

chiral vortical effect

chiral magnetic effect

Hattori, Yin (2016)

$$\begin{aligned} j_{(0)}^\mu &= n u^\mu, \\ j_{(1)}^\mu &= \xi\omega^\mu + \xi_B B^\mu, \\ j_{(2)}^\mu &= -\frac{\mu}{2\pi^2}(\varepsilon^2+\omega^2)u^\mu - \frac{1}{4\pi^2}(\varepsilon\cdot E + \boxed{\omega\cdot B)u^\mu} \\ &\quad -\frac{C}{12\pi^2}(E^2+B^2)u^\mu - \frac{1}{4\pi^2}\epsilon^{\mu\nu\rho\sigma}u_\nu E_\rho\omega_\sigma - \underbrace{\frac{C}{6\pi^2}\epsilon^{\mu\nu\rho\sigma}u_\nu E_\rho B_\sigma}_{\text{Hall term}} \end{aligned}$$

Yang-Gao-Liang-QW (2020)

Charge and axial charge currents up to $O(\hbar^2)$

- **The chiral charge currents are given by**

Yang-Gao-Liang-QW (2020)

$$
\begin{aligned}
j^{\mu}_{5,(0)} &= n_5 u^{\mu}, \\
j^{\mu}_{5,(1)} &= \xi_5 \omega^{\mu} + \xi_{B5} B^{\mu}, \\
j^{\mu}_{5,(2)} &= -\frac{\mu_5}{2\pi^2}(\varepsilon^2+\omega^2)u^{\mu} - \frac{C_5}{12\pi^2}(E^2+B^2)u^{\mu} - \overbrace{\frac{C_5}{6\pi^2}\epsilon^{\mu\nu\rho\sigma}u_{\nu}E_{\rho}B_{\sigma}}^{\text{anomalous Hall term}}
\end{aligned}
$$

Coefficients in the vector and axial vector currents

	j^{μ}		j^{μ}_5
n	$\frac{\mu}{3\pi^2}\left(\pi^2 T^2+\mu^2+3\mu_5^2\right)$	n_5	$\frac{\mu_5}{3\pi^2}\left(\pi^2 T^2+3\mu^2+\mu_5^2\right)$
ξ	$\mu\mu_5/\pi^2$	ξ_5	$\frac{1}{6\pi^2}\left[\pi^2 T^2+3(\mu^2+\mu_5^2)\right]$
ξ_B	$\mu_5/(2\pi^2)$	ξ_{B5}	$\mu/(2\pi^2)$
C	$\frac{1}{2}(C_+ + C_-)$	C_5	$\frac{1}{2}(C_+ - C_-)$
C	$-14\zeta'(-2)\mu/T^2,\ (\overline{\mu}_s \ll 1)$	C_5	$-14\zeta'(-2)\mu_5/T^2,\ (\overline{\mu}_s \ll 1)$
C	$\mu/(\mu^2-\mu_5^2),\ (\overline{\mu}_s \gg 1)$	C_5	$-\mu_5/(\mu^2-\mu_5^2),\ (\overline{\mu}_s \gg 1)$

Disentangling Wigner functions for chiral fermions (DWF theorem)

- Wigner function for chiral fermions have 8 real independent variables, since for each chirality it is a 2x2 Hermitian matrix that has 4 real variables, but there are 16 equations

16 entangled equations: 8 (right) + 8 (left)

Consistent ?

8 real variables: 4 (right) + 4 (left)

- In background EM fields, at any order of $\hbar$ or $O(\hbar^n)$, for each chirality, 8 entangled equations for 4 variables can be reduced to one evolution equation for a single distribution + one mass-shell equation

Chiral Kinetic Equation = 1 evolution equation for $f_s(x,p)$ with s =R, L + 1 equation for mass-shell condition

Redundancy removed!

Gao-Liang-QW-Wang (2018)

Key to the proof of DWF theorem

- **The chiral (R, L) component of Wigner function**

$$\mathscr{J}^{\mu} = (\underline{\mathscr{J}_0}, \underline{\boldsymbol{\mathscr{J}}})$$

time component: distribution

space component

- **One can prove that at $O(\hbar^n)$ can be expressed in terms of time parts at lower order $O(\hbar^i)$ with $i < n$**

$$\boldsymbol{\mathscr{J}}^{(n)} = F\left[\mathscr{J}_0^{(0)}, \mathscr{J}_0^{(1)}, \cdots, \mathscr{J}_0^{(n-1)}\right]$$

space component at $O(\hbar^n)$

time component at lower order

- **One can also introduce a frame 4-vector n^{μ} (time-like, $n^2 = 1$) to define the time and space parts in a covariant way**

Chiral Kinetic Equation

- **One can prove that the set of equations for covariant Wigner functions lead to the on-shell CKE at $O(\hbar)$**

on-shell distribution function

$$
\begin{aligned}
0 \;=\;& \left(1+\hbar s \underline{\boldsymbol{\Omega}_p}\cdot\mathbf{B}\right)\partial_t \underline{f(x,E_p,\mathbf{p})} \\
&+\left[\mathbf{v}+\hbar s(\mathbf{E}\times\underline{\boldsymbol{\Omega}_p})+\hbar s\frac{1}{2|\mathbf{p}|^2}\mathbf{B}\right]\cdot\boldsymbol{\nabla}_x \underline{f(x,E_p,\mathbf{p})} \\
&+\left[\widetilde{\mathbf{E}}+\mathbf{v}\times\mathbf{B}+\hbar s(\mathbf{E}\cdot\mathbf{B})\underline{\boldsymbol{\Omega}_p}\right]\cdot\boldsymbol{\nabla}_p \underline{f(x,E_p,\mathbf{p})}
\end{aligned}
$$

$\mathbf{v}\equiv\boldsymbol{\nabla}_p E_p^{(+)}$

$\widetilde{\mathbf{E}}\equiv\mathbf{E}-\boldsymbol{\nabla}_x E_p^{(+)}$

Berry curvature in momentum space

From effective theory: Stephanov-Yin (2012), Son-Yamamoto (2012, 2013); Manuel-Torres-Rincon (2014); Lin-Shukla (2019)
From covariant WF at $O(\hbar)$: Hidaka-Pu-Yang (2016); Gao-Liang-QW-Wang (2018); Huang-Shi-Jiang-Liao-Zhuang (2018)
From covariant CKE: Chen-Pu-QW-Wang (2013); Gao-Pang-QW (2017)

Frame decomposition of CME and CVE current

- Given $f(x,p)$ for chiral fermions, how to construct a charge current? A normal way is (for CVE)

particle

anti-particle

$$\boldsymbol{j}_{s,\omega} = \int \frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}}{|\mathbf{p}|}\left[f_{\mathrm{FD}}\left(|\mathbf{p}| - \mu_s + \underline{\Delta E_s}\right) + f_{\mathrm{FD}}\left(|\mathbf{p}| + \mu_s + \underline{\Delta E_s}\right)\right]$$

s =R,L

$$\Delta E_s = -s\hbar\frac{\mathbf{p}\cdot\boldsymbol{\omega}}{2|\mathbf{p}|}$$

spin-vorticity coupling energy

$$\approx \frac{1}{2}\hbar s\beta\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}}{|\mathbf{p}|^2}(\mathbf{p}\cdot\boldsymbol{\omega})\left[f_{\mathrm{FD}}^{+}(1-f_{\mathrm{FD}}^{+}) + f_{\mathrm{FD}}^{-}(1-f_{\mathrm{FD}}^{-})\right]$$

$$f_{\mathrm{FD}}^{\pm} = f_{\mathrm{FD}}\left(|\mathbf{p}| \mp \mu_s\right)$$

$$= \frac{1}{3}\times\mathrm{CVE}$$

- Where is the rest 2/3? → a puzzle. Need to get it back: it comes from "magnetization" current.

Chen-Son-Stephanov-Yee-Yin (2014); Kharzeev-Stephanov-Yee (2017)

Frame decomposition of CME and CVE current

- The method to solve space part from the time part makes the decomposition

$$
\begin{aligned}
\mathscr{J}^{\mu}_{(1)\omega} &= \underbrace{-p^{\mu}\frac{s}{2\,(n\cdot p)}n_{\alpha}p_{\gamma}\widetilde{\Omega}^{\alpha\gamma}f'_{(0)}\delta(p^2)}_{\text{(a) term}} \\
&\quad \underbrace{-\frac{s}{2n\cdot p}p^{\lambda}\epsilon^{\mu\nu\rho\sigma}n_{\nu}p_{\rho}\Omega_{\sigma\lambda}f'_{(0)}\delta(p^2)}_{\text{(b) term}} \\
&= -\frac{s}{2}\widetilde{\Omega}^{\mu\nu}p_{\nu}f'_{(0)}\delta(p^2)
\end{aligned}
$$

Total result without frame dependence

- (a) and (b) terms give (**puzzle solved!**)

$$
j^{\mu}_{s,\omega}(a) = \frac{1}{3}\times \mathrm{CVE}
$$

$$
\underbrace{j^{\mu}_{s,\omega}(b) = \frac{2}{3}\times \mathrm{CVE}}_{\text{"magnetization" current}}
$$

Similar for CME !

$$
\mathscr{J}^{\mu} = (\mathscr{J}\cdot n)n^{\mu} + \overline{\mathscr{J}}^{\mu}
$$

$$
\Big\Downarrow \quad n^{\mu} = (1,0,0,0)
$$

$$
\mathscr{J}^{\mu} = (\mathscr{J}_0, \boldsymbol{\mathscr{J}})
$$

Gao-Pang-QW, PRD (2019)

Chiral and spin kinetic theories in curved spacetime

- **Motivation: a framework studying chiral and spin transport under gravitational and non-inertial (e.g., rotational) forces**
- **Same $\hbar$ expansion for Wigner function but with replacement (horizontal lift of covariant derivative):** $\partial_\mu \to D_\mu = \nabla_\mu + \Gamma^\lambda_{\mu\nu} p_\lambda \partial^\nu_p$
- **The chiral kinetic equations:**

$$\delta\left(p^2 \mp \hbar F_{\alpha\beta}\Sigma_n^{\alpha\beta}\right)\left[p\cdot\Delta \pm \hbar\left(\frac{n_\mu \tilde{F}^{\mu\nu}}{p\cdot n} + \Delta_\mu \Sigma_n^{\mu\nu}\right)\Delta_\nu \pm \frac{\hbar}{2}\Sigma_n^{\mu\nu}\left(\nabla_\rho F_{\mu\nu} - p_\lambda R^\lambda{}_{\rho\mu\nu}\right)\partial_p^\rho\right](f \pm f_5) = \text{col.}$$

- **The spin kinetic equations:**

$$\Xi_{\mu\nu} \equiv \nabla_\rho F_{\mu\nu}\partial_p^\rho + [D_\mu, D_\nu]$$

$$\delta(p^2 - m^2 \mp \hbar\Sigma_S^{\alpha\beta}F_{\alpha\beta})\left\{\left[p^\mu\Delta_\mu \pm \frac{\hbar}{2}\Sigma_S^{\mu\nu}\Xi_{\mu\nu}\right](f \pm f_A) + \hbar f \Xi_{\mu\nu}\Sigma_S^{\mu\nu}\right\} = \text{col.}$$

$$\left\{f_A p\cdot\Delta\theta^\mu - F^{\mu\nu} f_A \theta_\nu + \theta^\mu\,(p\cdot\Delta f_A) - \frac{\hbar}{2m}\epsilon^{\mu\nu\rho\sigma}p_\sigma \Delta_\nu \Delta_\rho f\right\}\delta(p^2 - m^2) = \text{col}$$

$$\Sigma_S^{\mu\nu} = \frac{1}{2m}\epsilon^{\mu\nu\rho\sigma}\theta_\rho p_\sigma$$

$$\theta^\mu\theta_\mu = -1 \qquad p^\mu\theta_\mu = 0$$

Liu-Gao-Mameda-Huang (2018-2020)

Wigner functions for massive fermions

- **Wigner function (4x4 matrix) for massive fermions in background EM fields**

Heinz, PRL 51, 351 (1983);
Vasak-Gyulassy-Elze, Ann. Phys. 173, 462 (1987)

$$W_{\alpha\beta}(x,p) = \int \frac{d^4y}{(2\pi)^4} \exp\left(\frac{i}{\hbar} p \cdot y\right) \left\langle \psi_\beta^\dagger(x_2)\psi_\alpha(x_1)\right\rangle U(x_2, x_1)$$

- **Wigner function decomposition in 16 generators of Clifford algebra**

$$W = \frac{1}{4}\left[\mathscr{F} + i\gamma^5\mathscr{P} + \gamma^\mu\mathscr{V}_\mu + \gamma^5\gamma^\mu\mathscr{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathscr{S}_{\mu\nu}\right]$$

scalar p-scalar vector axial-vector tensor

$$j^\mu = \int d^4p\mathscr{V}^\mu, \qquad j_5^\mu = \int d^4p\mathscr{A}^\mu, \qquad T^{\mu\nu} = \int d^4p p^\mu\mathscr{V}^\nu$$

Vasak-Gyulassy-Elze, Ann. Phys. 173, 462 (1987);
Elze-Gyulassy-Vasak, Nucl. Phys. B 276, 706(1986);

WF solution for massive fermions (without collisions)

- **EOM for Wigner function**

$$(\gamma \cdot K - m)\, W(x,p) = 0$$

$$K^\mu = \Pi^\mu + \frac{1}{2} i\hbar \nabla^\mu$$

Equations for components

$$K \cdot \mathcal{V} - m\mathcal{F} = 0,$$
$$K \cdot \mathcal{A} + im\mathcal{P} = 0,$$
$$K_\mu \mathcal{F} + iK^\nu \mathcal{S}_{\nu\mu} - m\mathcal{V}_\mu = 0,$$
$$iK_\mu \mathcal{P} + \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} K^\nu \mathcal{S}^{\alpha\beta} + m\mathcal{A}_\mu = 0,$$
$$-iK_{[\mu}\mathcal{V}_{\nu]} - \epsilon_{\mu\nu\alpha\beta} K^\alpha \mathcal{A}^\beta - m\mathcal{S}_{\mu\nu} = 0,$$

- **Choose $\mathcal{F}$ and $\mathcal{S}^{\mu\nu}$ as independent components and solve EOM in perturbation method based on $\hbar$ expansion**

- **Zero-th order solution**

$$\mathcal{F}^{(0)}(x,p) = m\delta(p^2 - m^2) V^{(0)}(x,p),$$
$$\mathcal{P}^{(0)}(x,p) = 0,$$
$$\mathcal{V}^{(0)}_\mu(x,p) = p_\mu \delta(p^2 - m^2) V^{(0)}(x,p),$$
$$\mathcal{A}^{(0)}_\mu(x,p) = m n^{(0)}_\mu(x,p)\delta(p^2 - m^2) A^{(0)}(x,p),$$
$$\mathcal{S}^{(0)}_{\mu\nu}(x,p) = m\Sigma^{(0)}_{\mu\nu}(x,p)\delta(p^2 - m^2) A^{(0)}(x,p),$$

$$V^{(0)}(x,p) \equiv \frac{2}{(2\pi\hbar)^3}\sum_{es}\theta(ep^0) f_s^{(0)e}(x, e\mathbf{p})$$

$$A^{(0)}(x,p) \equiv \frac{2}{(2\pi\hbar)^3}\sum_{es} s\,\theta(ep^0) f_s^{(0)e}(x, e\mathbf{p})$$

$$n^{(0)}_\mu = -\frac{1}{2m}\epsilon_{\mu\nu\alpha\beta}p^\nu \Sigma^{(0)\alpha\beta}$$

Weickgenannt-Sheng-Speranza-QW-Rischke (2019)

WF solution for massive fermions (without collisions)

- **First order solution**

$$\mathcal{F}^{(1)} = m\left[V^{(1)}\delta(p^2-m^2) - \frac{1}{2}F^{\mu\nu}\Sigma^{(0)}_{\mu\nu}A^{(0)}\delta'(p^2-m^2)\right]$$

$$\mathcal{S}^{(1)}_{\mu\nu} = m[\bar{\Sigma}^{(1)}_{\mu\nu}\delta(p^2-m^2) - F_{\mu\nu}V^{(0)}\delta'(p^2-m^2)]$$

$$\mathcal{P}^{(1)} = \frac{1}{4m}\epsilon^{\mu\nu\alpha\beta}\nabla^{(0)}_{\mu}[p_\nu\Sigma^{(0)}_{\alpha\beta}A^{(0)}\delta(p^2-m^2)],$$

$$\mathcal{V}^{(1)}_{\mu} = \delta(p^2-m^2)\left[p_\mu V^{(1)} + \frac{1}{2}\nabla^{(0)\nu}\Sigma^{(0)}_{\mu\nu}A^{(0)}\right] - \left[\frac{1}{2}p_\mu F^{\alpha\beta}\Sigma^{(0)}_{\alpha\beta} + \Sigma^{(0)}_{\mu\nu}F^{\nu\alpha}p_\alpha\right]A^{(0)}\delta'(p^2-m^2),$$

$$\mathcal{A}^{(1)}_{\mu} = m\bar{n}^{(1)}_{\mu}\delta(p^2-m^2) + \tilde{F}_{\mu\nu}p^\nu V^{(0)}\delta'(p^2-m^2),$$

$$\bar{n}^{(1)}_{\mu} \equiv -\frac{1}{2m}\epsilon_{\mu\nu\alpha\beta}p^\nu\bar{\Sigma}^{(1)\alpha\beta}$$

$V^{(1)}$ and $\bar{\Sigma}^{(1)}_{\mu\nu}$ are determined by kinetic equations

Weickgenannt-Sheng-Speranza-QW-Rischke (2019)

Kinetic equation for massive fermions (without collisions)

- **Kinetic equation to $O(\hbar)$**

$$0=\delta(p^2-m^2)\left[p\cdot\nabla^{(0)}\underline{V}+\frac{\hbar}{4}(\partial_x^\alpha F^{\mu\nu})\partial_{p\alpha}\underline{\bar{\Sigma}_{\mu\nu}}\right]$$
$$-\frac{\hbar}{2}\delta'(p^2-m^2)F^{\alpha\beta}p\cdot\nabla^{(0)}\underline{\bar{\Sigma}_{\alpha\beta}}+\mathcal{O}(\hbar^2),$$

$$0=\delta(p^2-m^2)\left[p\cdot\nabla^{(0)}\underline{\bar{\Sigma}_{\mu\nu}}-F^{\alpha}_{\ [\mu}\underline{\bar{\Sigma}_{\nu]\alpha}}+\frac{\hbar}{2}(\partial_{x\alpha}F_{\mu\nu})\partial_p^\alpha \underline{V}\right]$$
$$-\hbar\delta'(p^2-m^2)F_{\mu\nu}p\cdot\nabla^{(0)}\underline{V}+\mathcal{O}(\hbar^2).$$

$$V\equiv V^{(0)}+\hbar V^{(1)}+\mathcal{O}(\hbar^2),$$
$$\bar{\Sigma}^{\mu\nu}\equiv\Sigma^{(0)\mu\nu}A^{(0)}+\hbar\bar{\Sigma}^{(1)\mu\nu}+\mathcal{O}(\hbar^2).$$

under a constraint for V and $\bar{\Sigma}_{\mu\nu}$

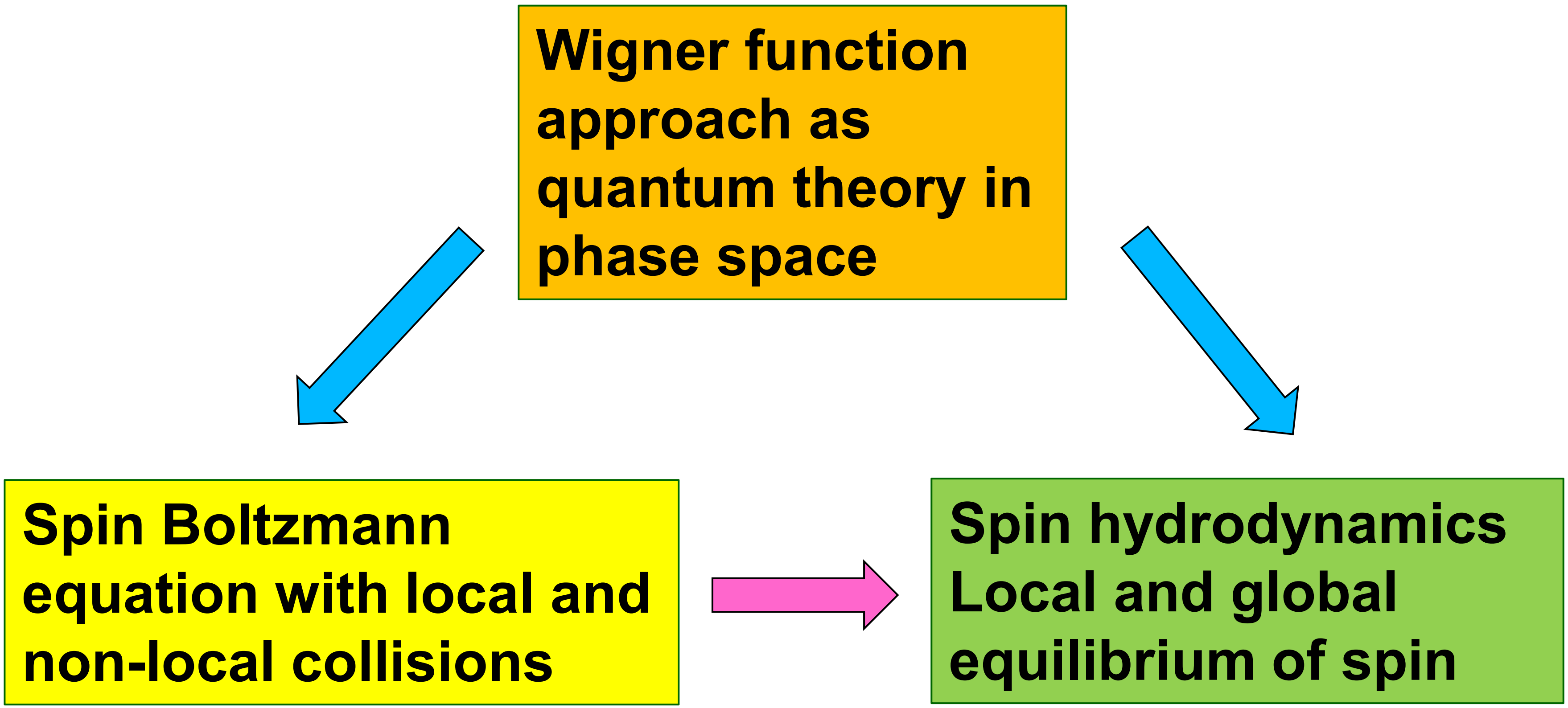
$$p^\nu\bar{\Sigma}_{\mu\nu}\delta(p^2-m^2)=\frac{\hbar}{2}\delta(p^2-m^2)\nabla_\mu^{(0)}V+\mathcal{O}(\hbar^2)$$

Weickgenannt-Sheng-Speranza-QW-Rischke (2019)

- **One can also choose other components as independent ones, e.g. $\mathcal{V}_\mu$ and $\mathcal{A}_\mu$, for WF of massive fermions**

Gao-Liang (2019); Hattori-Hidaka-Yang (2019); Wang-Guo-Shi-Zhuang (2019)

Summary


Wigner function approach as quantum theory in phase space
Spin Boltzmann equation with local and non-local collisions
Spin hydrodynamics Local and global equilibrium of spin