Two loop thermodynamic equation of state at finite chemical potential

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Abstract

Two loop thermodynamic equation of state (EOS) at finite chemical potential is studied at one stable droplet size. Due to the presence of the chemical potential in two loop correction, formation of bubble/droplet of quark gluon plasma (QGP) is modified, and at this particular stable droplet the thermodynamic parameters such as pressure, energy density, interaction measurement,entropy, specific heat and speed of sound are determined and compared with the relevant lattice data. The effect of chemical potential modifies the earlier results in close proximity with lattice and it is very good to represent the equation of state of forming quark gluon plasma formation. It means that the effect of chemical potential in two loop correction can signify the presence of large number of charge particles in the earlier universe formation indicating the high density nuclear matter.

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1 Introduction

Quantum chromodynamics (QCD) study says that there is a strong interactions of coloured free quarks and gluons. The theory also predicts that there is a phase transition from a free matter of quarks and gluons to another matter of bound quarks of hadrons. Before its change of phase, the matter of free quarks and gluons is called quark-gluon plasma (QGP). The QGP matter stays for a short period of time. Due to the short of existence, it is very difficult to extract and explore the information about the matter. However this short period of time can be found in two different aspects. One is obtained at a very high temperature and another one is found at a very high nuclear density. Experimentally the matter is created by the experiments at very high energy scale, which can be obtained by colliding two heavy nuclei accelerating in opposite direction and smash each other [1]. These experimental facilities are established at two places of the world. One is at BNL, New York. In this experiment two Au-Au nuclei are collided moving at relativistic limits known as relativistic heavy-ion collider physics (RHIC). The another one is LHC at CERN, Geneva. Here, in LHC two relativistic movements of Pb-Pb collisions are performed to see the consequence of the collision. So mini big bang called quark-gluon plasma has been created by these two experiments two decades ago. It indicates that the two experiments use high energy scale to create the short life matter but it's study is still continuing to have a depth confirmation about the existence of QGP as believed to be occurred just after the big Bang. [2].

BNL [3] and LHC [4, 5, 6] Having these two experiments it could not be studied about the matter at a very high density region. So two additional experimental facilities are established to explore about the dense matter region. These laboratories are specially built up because of this reason and both the experiments are focused on compact baryon matter (CBM). One experiment is at Darmstadt in Germany called as FAIR experiment and another one is at Dubna, Russia called as NICA [7]. These two laboratory studies are mainly specially based on dense baryonic matter called as compact baryonic matter (CBM) [8]. From all these experiments, it is understood that QGP has a lot of issues to be identified. One of the most important issue is to find the critical point on phase structure diagram at both the extreme cases of temperature and baryon chemical potential. So these experiments try to predict the critical point on phase diagram by performing experiments of increased energy scale. However there is a prediction of crossover phase transition at high

temperature and at low chemical potential, and first order phase transition at low temperature and high chemical potential. yo predict these observations, the article is written with consideration of two loop correction in the interacting potential of participating particles at finite chemical potential. Due to finite chemical potential, the participant particles are affected by temperature and chemical potentials. So,equation of state (EOS) is calculated using thermal and chemical potential dependent mass.

So the paper is on the calculation of equation of state (EOS) at the particular value of quark and gluon flow parameters at finite chemical potential. In the calculation it is a little modification in the interaction potential one loop extending to two loop correction with chemical potential [13, 14].

Briefly, the revised calculation of free energy created through density of state, that is created through a potential model. The potential is obtained through two loop correction and with finite chemical potential. Once free energy creation is done, and then it is further calculated equation of state like pressure, energy density, interaction measurement entropy, specific heat and speed of sound.

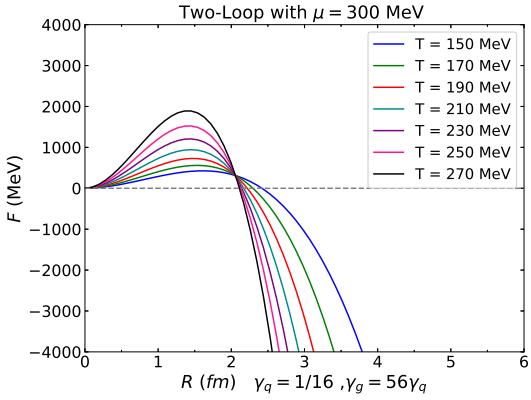

Figure 1: Free energy vs. R for different temperatures at $\gamma_q = 1/8$, $\gamma_g = 56\gamma_q$ for $\mu = 300\ MeV$.

2 Two loop thermodynamic free energy at finite chemical potential

The partition function of any system,knowing temperature, chemical potential and number density is given in most of literature as [21, 22]:

$$Z(T, \mu, V) = Tr\{exp[-\beta(\hat{H} - \mu\hat{N})]\} \tag{1}$$

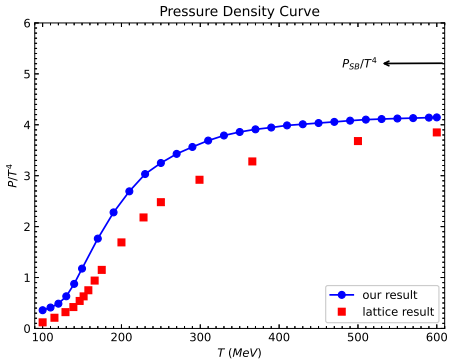

Figure 2: Pressure vs. T at $\gamma_q = 1/8$, $\gamma_g = 56\gamma_q$ for $\mu = 300\ MeV$.

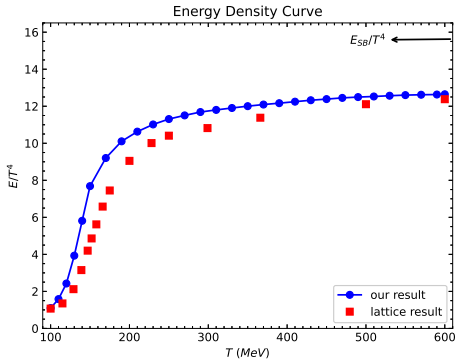

Figure 3: Energy density vs. T at $\gamma_q = 1/8$, $\gamma_g = 56\gamma_q$ for $\mu = 300\ MeV$.

where μ is chemical potential, $\hat{N}$ as number density. $\hat{H}$ is the Hamiltonian of the system and $\beta = \frac{1}{T}$. The Hamiltonian $\hat{H}$ of the system is found to be as:

$$\hat{H}(p,T) = p + m_0^2/2p - [m_0^2 - m(T,\mu)^2]/2p \tag{2}$$

where, the thermal and chemical potential mass is defined as:

$$m^2(p,\mu) = 2\gamma^2 g^2(p,\mu)(T^2 + \mu^2/\pi^2)[1 + g^2(p,\mu)a_1 + g^4(p,\mu)a_2] \tag{3}$$

The partition function in above relation is connected through the Gibbs free energy. So the Gibbs free energy with inclusion of two loop correction factor at finite chemical potential is given below [23, 24]:

$$F_i = \pm T g_i \int \rho_{q,g}(p,\mu) \ln[1 + \eta e^{-(\sqrt{m_i^2+p^2}-\mu)/T}]dp\ , \tag{4}$$

where g_i is degree of freedom of color and particle-antiparticle for quarks and gluons.

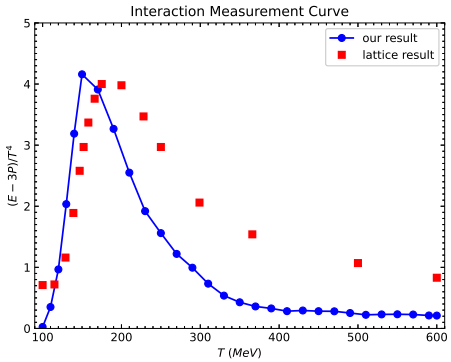

Figure 4: Interaction measurement vs. T at $\gamma_q = 1/8$, $\gamma_g = 56\gamma_q$ for $\mu = 300\ MeV$.

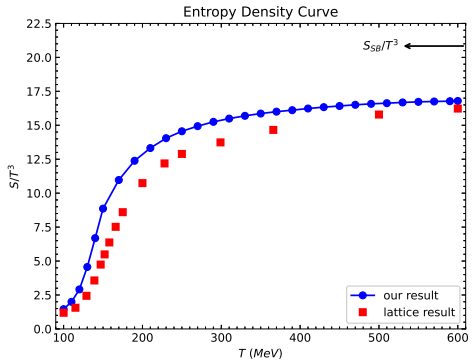

Figure 5: Entropy density vs. T at $\gamma_q = 1/8$, $\gamma_g = 56\gamma_q$ for $\mu = 300\ MeV$.

$\rho_{q,g}$ is phase space density of states after modification with two loop correction and it is derived after a thorough calculation with the Thomas-Fermi model. It is now represented as [25]:

$$\rho_{q,g}(p,\mu) = \frac{\nu}{\pi^2} g'(T,\mu) g^6(p,\mu) B, \tag{5}$$

where ν is the volume occupied by the QGP.

$g'(T,\mu) = [\frac{\gamma_{q,g}^3 [T^2+\mu^2/\pi^2]}{2}]^3$

is coupling constant. $\gamma_{q,g}$ is quark and gluon parametrization factors where $\gamma_q = 1/8$ and $\gamma_g = 56\ \gamma_q$ used as the value of stable droplet of QGP. $g^2(p,\mu) = 4\pi\alpha_s(p,\mu)$ is the varying coupling constant expressed in terms of
$\alpha_s(p,\mu) = \frac{4\pi}{29\ln[1+(p^2+\mu^2)/\Lambda^2]}$

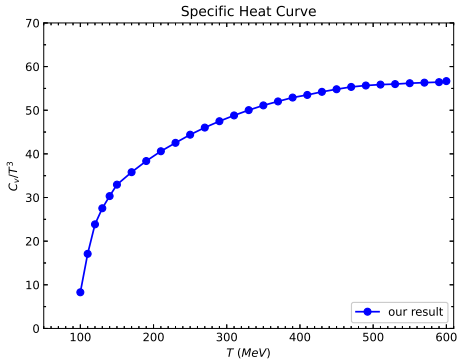

Figure 6: Specific heat vs. T at $\gamma_q = 1/8$, $\gamma_g = 56\gamma_q$ for $\mu = 300\ MeV$.

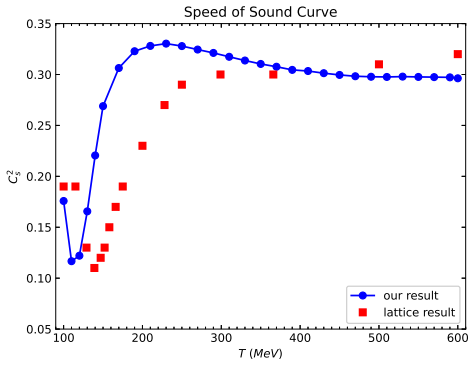

Figure 7: Speed of sound vs. T at $\gamma_q = 1/8$, $\gamma_g = 56\gamma_q$ for $\mu = 300\ MeV$.

where Λ is quantum chromodynamics parameter. B in above eqn.(4) is connected with varying coupling with two loop coefficient as:

$$B = \{1+\frac{\alpha_s(p,\mu)a_1}{\pi}+\frac{\alpha_s^2(p,\mu)a_2}{\pi^2}\}^2[\frac{(1+\alpha_s(p,\mu)a_1/\pi+\alpha_s^2(p.\mu)a_2/\pi^2)}{p^4}+\frac{2(1+2\alpha_s(p,\mu)a_1/\pi)}{p^2(p^2+\Lambda^2)\ln(1+\frac{p^2}{\Lambda^2})} \tag{6}$$

Incorporating all these factors, we determine the dynamics of QGP flow and the changes produced in the QGP-hadrons phase transitions. The coefficient $a_1 = 2.5833 - 0.2778n_l$, and $a_2 = 28.5468 - 4.1471n_l + 0.0772n_l^2$ with n_l as number of light quarks elements [26, 27]. Normally these light quarks are u, d and s quarks. In addition to these fermion and bosonic energies, there is an additional inter-facial energy which replaces the Bag energy constant and moreover this inter-facial energy has taken care of the

hydro-dynamical effects. It is defined as:

$$F_{interface} = \frac{\gamma T R^2}{4} \int p^2 \delta(p - T) dp, \tag{7}$$

in which γ is root mean square value of quark γ_q and gluon flow parameter γ_g. R is QGP bubble/droplet size and the hadronic free energy contribution with the corresponding degeneracy factor of g is given as [28]:

$$F_h = (gT/2\pi^2)\nu \int_0^\infty p^2 \ln[1 - e^{-(\sqrt{m_h^2+p^2}-\mu/T)}] dp. \tag{8}$$

where, m_h is the corresponding mass of hadronic particles used in the evaluation of the free energies. We can thus compute the total modified free energy F_{total} as,

$$F_{total} = \sum_i F_i \ + \ F_{interface} \ + \ F_h, \tag{9}$$

where i stands for u, d and s quark and gluon.

3 Two loop thermodynamic equation of state at finite chemical potential

The bulk thermodynamic quantities like equation of state, pressure, energy density, interaction measurement, entropy density, specific heat and speed of sound are evaluated to represent the whole collective behaviour of the QCD phase structure at finite chemical potential. These thermodynamic quantities provide the entire QCD phase structure under two loop correction at finite chemical potential. The evaluation of these bulk quantities are performed through the Gibbs free energy related with partition function of the grand canonical ensemble. It also gives the detailed information about QGP fireball with different flow parametrization values. Now, pressure is evaluated from the free energy using the standard thermodynamic formula and it is obtained in terms of density of state as: [29, 30]

$$\begin{aligned} P(T,\mu) &= -\eta T^2 \sum_i g_i \int \rho'_{two}(p) p^2 w(p) dp \\ &+ \frac{3\gamma T}{16\pi R} \int p^2 \delta(p - T) dp, \end{aligned} \tag{10}$$

where, $w(p) = \ln[1+\eta e^{-(\sqrt{m_i^2+p^2}-\mu)/T}]$ and ρ'_{two} is density of state independent of volume. g_i is degeneracy factor having different values of the corresponding particles viz quarks, gluons and hadrons.

We further calculate the energy density using the definition of thermodynamic potential as the partial derivative of it with the temperature. The energy density is equivalent using the standard relation through the free energy and it is defined as:

$$\epsilon = T^2 \frac{\partial \Omega}{\partial T} \tag{11}$$

After the calculation of the energy density we further look at the interaction measurement which is given by the relation:
$(\epsilon - 3p)/T^4$ This evaluation is done through the results of energy density and pressure of the system. Then we further evaluate the entropy density by the following relation calculated through the first order derivative of the free energy with the temperature and it is found as.

$$\begin{aligned} S &= \eta \sum_i g_i \int \rho_{two}(p) p^2 w(p) dp \\ &+ \frac{\eta}{T} \sum_i g_i \int \frac{\rho_{two}(p) p^2 \sqrt{(}m^2 + p^2 - \mu) dp}{[1 + \eta e^{-(\sqrt{m_i^2 + p^2} - \mu)/T}]} \\ &- \frac{3V\gamma T^3}{16\pi R}. \end{aligned} \tag{12}$$

Similarly we differentiate the entropy with temperature and multiplied by the temperature to get the specific heat and obtained as:

$$\begin{aligned} C_v &= \frac{1}{T^2} \sum_i g_i \int \frac{\rho_{two}(p) p^2 (m^2 + p^2 - \mu) x(p) dp}{[1 + \eta e^{-(\sqrt{m_i^2 + p^2} - \mu)/T}]^2} \\ &- \frac{V\gamma T^3}{16\pi R} \end{aligned} \tag{13}$$

where, $x(p) = [e^{-(\sqrt{m_i^2 + p^2} - \mu)/T}]$. Thus the entropy and the specific heat are evaluated at finite chemical potential with one loop correction having suitable choice of flow parameter of γ_q and gluon parameter γ_g, particularly as $\gamma_q = 1/8$ and $\gamma_g = 16\gamma_q$ at which the most stable droplet QGP formation are found. The calculations of these parameters determine the equilibrating state and disordered condensate states of the system. So the behavior of entropy and specific heat with temperature indicate the possible feature of phase transition as first or second order transition. Moreover many authors mentioned in many literature that the first and second order

transition is determined on the basis of first and second derivative of the Gibbs potential energy. In addition to these thermodynamic parameters, we calculate speed of sound using the ratio of entropy density and specific heat [31, 32, 33]. This parameter also indicates the dynamics of fluid flow.

$$C_s^2 = \frac{S/T^3}{C_v/T^3}. \tag{14}$$

Besides these thermodynamic properties, we further calculate number density, and susceptibility through the following relations obtained from the free energy. The relations of number density and susceptibility are found with first and second order derivatives of the free energy with respect to chemical potential.

Thus the quark number density and the susceptibility are obtained after the analytic calculation using the standard formulae [34, 35, 36].

$$n = -\eta \sum_i g_i \int \frac{\rho_{two}(p) p^2 dp}{[1 + \eta e^{(\sqrt{m_i^2+p^2}-\mu)/T}]} \tag{15}$$

$$\chi = -\frac{1}{T} \sum_i g_i \int \frac{\rho_{two}(p) p^2 e^{(\sqrt{m_i^2+p^2}-\mu)/T} dp}{[1 + \eta e^{(\sqrt{m_i^2+p^2}-\mu)/T}]^2} \tag{16}$$

The evaluated values represent the behavior of number density and susceptibility in figures.

4 Results:

In this brief paper, we calculate the equation of state like pressure, energy density, interaction measurement, entropy density, specific heat, speed of sound and other parameters. So in Figs.1 to 8, the bulk thermodynamic parameters are represented. In Fig.1, there is pressure versus temperature and the behaviour of the pressure is absolutely in the range of the lattice output [37]. It says that the QGP droplet can be formed at finite temperature and at finite chemical potential with temperature and chemical potential dependent quark mass.

Similarly, in Fig.2 we check the energy density by differentiating the free energy and the behaviour of the energy density also follows the normal behaviour as predicted by the lattice. It implies that the equation of state of QGP at finite chemical potential with this

one loop correction is appropriately fine and the model of such one loop correction can be switched on for the calculation of QGP fluid dynamics. For the confirmation of these information, we again plot the interaction measurement which is defined by $(\epsilon - 3P)/T^4$. The plot is lying in between the amplitude of pressure and energy density which is within the range of predicted lattice calculation [38].

Further in Fig.4**,** we look at the entropy density versus the temperature. The entropy density is quite understood that it can represent the equilibrium state of the system. The plot is however compared with the lattice available data [39]. From the comparison it indicates that the entropy of our model of one loop correction at finite chemical potential is exactly following the data of the lattice with small deviation in the lower temperature and completely follows in the higher temperature. This is one evidence that the model fits well in the construction of QGP droplet formation with one loop correction at finite chemical potential.

Again we check in the specific heat calculation in furtherance of our bulk thermodynamic parameters. The specific heat calculation also follows the similar pattern with our earlier values of specific heat**which is shown in Fig.**5**.** The pattern also indicates that there is an agreement of the present model with the other theoretical predictions. After obtaining the entropy and specific heat results we use these two results in evaluating the speed of sound. The speed of sound is obtained as the ratio of entropy density to the specific heat. The Fig.6 shows the corresponding output of speed of sound. In the lower temperature, our result is slightly high in comparison with the lattice result and in the higher temperature, the speed of sound is in conformity with the presently existing lattice data. This implies that the model has a little short at the lower temperature due to the dominance of chemical potential over the lower temperature. So the formation of QGP droplet is highly appreciable with this one loop correction at finite chemical potential.

At last we further calculate the number density and susceptibility which can give the details about the number fluctuations.**These are demonstrated in Figs.**$(7-8)$**.** The number density increases rapidly at the lower temperature as the temperature increases up to 300 MeV and it starts saturation with the temperature reaches around $T = 300$ MeV. This is also one idea which explains about the number fluctuation in QGP formation.

Further we plot the susceptibility produced by the system. In this **figure** 8**,** it represents that susceptibility also increases exponentially and this exponential increasing is saturated when the temperature reached around $T = 400$ MeV. It indicates that the

magnetic effects of the system has a significant amount of effect in the formation of the QGP bubbles/droplets. However, the QGP fluid has in reality the magnetic effects which are predicted by the ongoing experiments. So these thermodynamic parameters can explain at least some of the important features of QGP dynamics and QCD phase structure.

So we can conclude that the extensive work of our earlier calculation in one loop correction with the incorporation of chemical potential can still produce the stable droplet in the flow of QGP dynamics and the equation of state at this stable droplet can be determined.

5 conclusion:

We summarize from these numerical results that due to the presence of chemical potential and one loop correction in the mean field potential, the stability of droplets is obtained in the range of quark and gluon flow parameters as $\gamma_q = 1/8$ and $\gamma_g = (12-16)\gamma_q$. So, we can further conclude that QGP fluid dynamics can be controlled by some sorts of hydro-dynamical parameters called quark and gluon flow parameters to get the stable droplets of the fluid. Depending on the stability of the droplet we studied other thermodynamic properties of QGP like pressure, energy density and other thermodynamic properties. The properties show very much similar in pressure, energy density and interaction measurements of lattice outputs. The results of entropy density are almost the lattice outputs. In the case of specific heat there is a little high in the higher temperature zone. Looking the speed of sound, there is a little difference in lower temperature and at higher the speed of sound is a little from the lattice. Therefore it can be concluded that the presence of chemical potential in the model put forward a closed results with the lattice. It is a good significant in representing the equation of state of the quark gluon plasma with the chemical potential.

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