

Basics of spin physics in relativistic quantum theory

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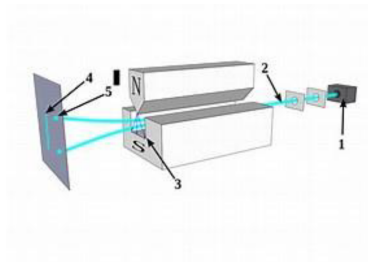
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2025.11.3

- Introduction: 100 years of spin
- Spin in non-relativistic quantum theory
- Spin in relativistic quantum theory
- Spin dependent distributions and Wigner functions
- References: [S. Weinberg, Phys.Rev. 133, B1318 (1964); S. Weinberg, The quantum theory of fields I, Cambridge University Press (1995); P. Mousa, R. Stora, Angular analysis of elementary particle reactions, in Proceedings of 7-th Summer Institute for Theoretical Physics, 37-69; F. Becattini, et. al., Ann. Phys. 338, 32 (2013); Xin-li Sheng, et.al., Phys. Rev. D104, 016029 (2021).]

100 years of spin: Stern-Gelach experiment

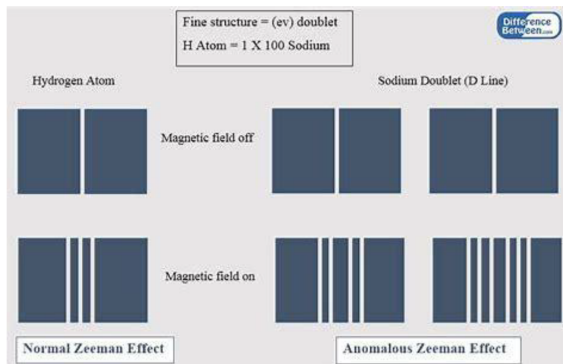
- The classical picture of a particle's spin fails due to the problem of exceeding speed of light.
- **Stern-Gerlach experiment (1922):** first observation of two discrete quantum states of silver atom with μ_B in non-homogeneous B field



Otto Stern, Nobel prize in Physics 1943

100 years of spin: Anomalous Zeeman effect

- Zeeman effect (1896), Anomalous Zeeman effect (1920s):** quantization of orbital angular momentum and spin.



Pieter Zeeman, Nobel prize in Physics 1903

100 years of spin: Fourth quantum number

- **Fourth quantum number by Wolfgang Pauli (1924):** to explain anomalous Zeeman effect, which takes only two values.
- **Concept of electron spin by Ralph Kronig (1925):** can explain even splitting of alkali spectra (over-estimated by factor 2), but opposed by Pauli and Bohr, not published



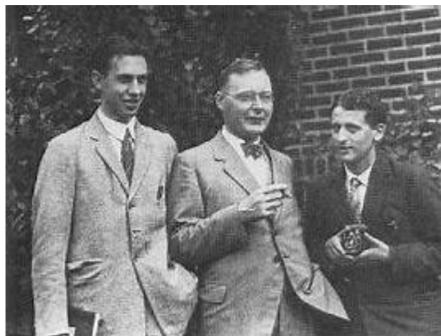
Wolfgang Pauli,
Nobel prize in Physics 1945



Ralph Kronig

100 years of spin: Discovery of electron spin

- **Electron spin by Uhlenbeck and Goudsmit (1925):**



G.E. Uhlenbeck and S. Goudsmit,
Naturwissenschaften 13 (1925)
953.

A subsequent publication by the
same authors, *Nature* 117 (1926)
264.

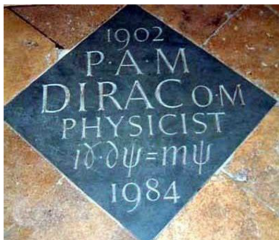
George Uhlenbeck, Samuel Goudsmit

100 years of spin: Dirac equation for relativistic particles

- **Pauli's non-relativistic theory for electron spin (1927):** Schroedinger equation for particle with spin-1/2, Pauli spinor, Pauli matrices (dimension 2)

$$\hat{H}|\psi\rangle = \left[\frac{1}{2m} [(\mathbf{p} - q\mathbf{A})^2 - q\hbar\boldsymbol{\sigma} \cdot \mathbf{B}] + q\phi \right] |\psi\rangle = i\hbar \frac{\partial}{\partial t} |\psi\rangle$$

- **Dirac equation (1928):** relativistic extension of Pauli's theory, Dirac spinor, Dirac matrices (dimension 4)



Paul Dirac
Nobel Prize
in physics
1933

100 years of spin: Dirac equation for relativistic particles

Dirac equation for spin-1/2 particle in a central potential $V(r) \sim -C/r$

$$\begin{aligned} (\boldsymbol{\alpha} \cdot \hat{\mathbf{P}} + \beta m + V) \begin{pmatrix} \chi \\ \psi \end{pmatrix} &= E \begin{pmatrix} \chi \\ \psi \end{pmatrix} & \hat{\mathbf{P}} &= -i\vec{\nabla} \\ \begin{pmatrix} E - V - m & -\boldsymbol{\sigma} \cdot \hat{\mathbf{P}} \\ -\boldsymbol{\sigma} \cdot \hat{\mathbf{P}} & E - V + m \end{pmatrix} \begin{pmatrix} \chi \\ \psi \end{pmatrix} &= 0 & \chi \text{ and } \psi &\text{ are Pauli spinors} \end{aligned}$$

Expressing ψ (small component) in terms of χ (large component) in non-relativistic approximation, we obtain spin-orbit coupling term $H_{S.O.}$.

$$\begin{aligned} E_S \chi &= \left[V + (\boldsymbol{\sigma} \cdot \hat{\mathbf{P}}) \frac{1}{E - V + m} (\boldsymbol{\sigma} \cdot \hat{\mathbf{P}}) \right] \chi & \frac{1}{E - V + m} &\approx \frac{1}{2m} \left(1 - \frac{E_S - V}{2m} \right) = \frac{1}{2m} - \frac{E_S - V}{4m^2} \\ &\approx \left[\frac{\hat{\mathbf{P}}^2}{2m} + V - (\boldsymbol{\sigma} \cdot \hat{\mathbf{P}}) \frac{E_S - V}{4m^2} (\boldsymbol{\sigma} \cdot \hat{\mathbf{P}}) \right] \chi & E_S &\equiv E - m \\ &= \left[\frac{\hat{\mathbf{P}}^2}{2m} + V - \frac{\hat{\mathbf{P}}^4}{8m^3} - i \frac{(\boldsymbol{\sigma} \cdot \hat{\mathbf{P}}) \times [\hat{\mathbf{P}}, V]}{4m^2} - \frac{\hat{\mathbf{P}} \cdot [\hat{\mathbf{P}}, V]}{4m^2} \right] \chi \\ H_{S.O.} &= -i \frac{(\boldsymbol{\sigma} \cdot \hat{\mathbf{P}}) \times [\hat{\mathbf{P}}, V]}{4m^2} = C \frac{\boldsymbol{\sigma} \cdot (\mathbf{r} \times \hat{\mathbf{P}})}{4m^2 r^3} & [\hat{\mathbf{P}}, V] &= [-i\nabla, V] = -iC \frac{\mathbf{r}}{r^3} \end{aligned}$$

Effective Hamiltonian

spin-orbit coupling is a relativistic effect and can be derived from Dirac equation !

Energy in rotating frame

In the fixed inertial frame S , the frame S' is rotating about a fixed axis (e.g., the z -axis) S with a constant angular velocity $\boldsymbol{\omega}$.

A particle with the mass m , with a position vector $\mathbf{r}'$ and velocity $\mathbf{v}'$ in frame S' , its absolute velocity $\mathbf{v}$ in frame S is

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' \quad (1)$$

The kinetic energy T of the particle in S is

$$\begin{aligned} T &= \frac{1}{2} m \mathbf{v} \cdot \mathbf{v} = \frac{1}{2} m (\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}') \cdot (\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}') \\ &= \frac{1}{2} m v'^2 + m \mathbf{v}' \cdot (\boldsymbol{\omega} \times \mathbf{r}') + \frac{1}{2} m (\boldsymbol{\omega} \times \mathbf{r}')^2 \end{aligned} \quad (2)$$

Energy in rotating frame

The Lagrangian is $L = T - V$, and the generalized momentum is

$$\mathbf{p}' = \frac{\partial L}{\partial \mathbf{v}'} = \frac{\partial T}{\partial \mathbf{v}'} = m(\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}') = m\mathbf{v} \quad (3)$$

The Hamiltonian is

$$\begin{aligned} H_{\omega} &= \mathbf{p}' \cdot \mathbf{v}' - L = \frac{1}{2}mv'^2 - \frac{1}{2}m(\boldsymbol{\omega} \times \mathbf{r}')^2 + V \\ &= \frac{1}{2m}(\mathbf{p}' - m\boldsymbol{\omega} \times \mathbf{r}')^2 - \frac{1}{2}m(\boldsymbol{\omega} \times \mathbf{r}')^2 + V \\ &= \frac{1}{2m}\mathbf{p}'^2 - \mathbf{p}' \cdot (\boldsymbol{\omega} \times \mathbf{r}') + V = \frac{1}{2m}\mathbf{p}'^2 + V - \boldsymbol{\omega} \cdot \mathbf{L}' \end{aligned} \quad (4)$$

where $\mathbf{L}' = \mathbf{r}' \times \mathbf{p}'$.

Energy in magnetic field

In classical mechanics, the magnetic moment of a particle in an external magnetic field can be derived as follows

$$\begin{aligned}T &= \frac{(\mathbf{p} - q\mathbf{A})^2}{2m} = \frac{\mathbf{p}^2}{2m} - \frac{q}{m}\mathbf{p} \cdot \mathbf{A} + \frac{q^2}{2m}\mathbf{A}^2 \\ -\frac{q}{m}\mathbf{p} \cdot \mathbf{A} &= \frac{q}{2m}\mathbf{p} \cdot (\mathbf{r} \times \mathbf{B}) = -\frac{q}{2m}\mathbf{L} \cdot \mathbf{B} = -\boldsymbol{\mu}_L \cdot \mathbf{B} \\ H_B &= \frac{\mathbf{p}^2}{2m} + \frac{q^2}{2m}\mathbf{A}^2 + V - \boldsymbol{\mu}_L \cdot \mathbf{B}\end{aligned}\tag{5}$$

Here we choose $\mathbf{A} = -\frac{1}{2}\mathbf{r} \times \mathbf{B}$ (Coulomb gauge condition is satisfied $\nabla \cdot \mathbf{A} = 0$) and orbital magnetic moment $\boldsymbol{\mu}_L = \frac{q}{2m}\mathbf{L}$ with $\mathbf{L} = \mathbf{r} \times \mathbf{p}$.

Including spin

We can include the spin contribution to the interaction energy

$$\begin{aligned} H_{\omega} &= \frac{1}{2m} \mathbf{p}^2 + V - (\mathbf{L} + \mathbf{S}) \cdot \boldsymbol{\omega} \\ H_B &= \frac{\mathbf{p}^2}{2m} + \frac{q^2}{2m} \mathbf{A}^2 + V - (\boldsymbol{\mu}_L + \boldsymbol{\mu}_S) \cdot \mathbf{B} \end{aligned} \quad (6)$$

where we have suppressed the prime symbol in the rotating frame, and $\boldsymbol{\mu}_S = g \frac{q}{2m} \mathbf{S}$ is the spin magnetic moment.

Spin in non-relativistic quantum theory

Spin in non-relativistic quantum theory

Spin in non-relativistic thermal systems

In non-relativistic quantum-mechanics, the mean spin vector is defined as:

$$\mathbf{S} = \langle \hat{\mathbf{S}} \rangle = \text{Tr} (\hat{\rho} \hat{\mathbf{S}}) \quad (7)$$

where $\hat{\rho}$ is the density operator of the particle under consideration and $\hat{\mathbf{S}}$ is the spin operator. The polarization vector is defined as the mean value of the spin operator normalized to the spin quantum number of the particle:

$$\mathbf{P} = \frac{1}{S} \langle \hat{\mathbf{S}} \rangle \quad (8)$$

so that its maximal value is 1, that is $|\mathbf{P}| \leq 1$.

Spin in non-relativistic thermal systems

Consider a non-relativistic particle at equilibrium in a thermal bath at temperature T in a rotating vessel at an angular velocity ω , the density operator is

$$\hat{\rho} = \frac{1}{Z} \exp \left[-\beta \hat{H} + \beta \mu \hat{Q} + \beta \boldsymbol{\omega} \cdot \hat{\mathbf{J}} + \beta \hat{\boldsymbol{\mu}}_B \cdot \mathbf{B} \right] \quad (9)$$

where $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$, $\hat{Q}$ is a conserved charge with μ being the corresponding chemical potential, and $\mathbf{B}$ is a constant and uniform external magnetic field with $\hat{\boldsymbol{\mu}}_B = \mu_B \hat{\mathbf{S}}/S$ being the magnetic moment.

Spin in non-relativistic thermal systems

If the constant angular velocity ω , as well as the constant magnetic field $\mathbf{B}$ are parallel, the above density operator can be diagonalized in the basis of eigenvectors of the spin operator component parallel to ω , $\hat{\mathbf{S}} \cdot \omega$, thereby giving rise to a probability distribution for its different eigenvalues m . The different probabilities read

$$w[T, B, \omega](m) = \frac{\exp \left[\beta \left(\frac{\mu_B B}{S} + \omega \right) m \right]}{\sum_{m=-S}^S \exp \left[\beta \left(\frac{\mu_B B}{S} + \omega \right) m \right]} \quad (10)$$

The distribution Eq. (10) may now be used to estimate the spin vector in Eq. (7).

Spin in non-relativistic thermal systems

For the simpler case with $B = 0$, the spin vector along ω is

$$\begin{aligned}\mathbf{S} &= \hat{\omega} \frac{\sum_{m=-S}^S m \exp(\beta \omega m)}{\sum_{m=-S}^S \exp(\beta \omega m)} = \hat{\omega} \frac{\partial}{\partial(\beta \omega)} \ln \left[\sum_{m=-S}^S \exp(\beta \omega m) \right] \\ &= \hat{\omega} \frac{\partial}{\partial(\beta \omega)} \ln \{ \exp(-\beta \omega S) [1 + \exp(\beta \omega) + \cdots + \exp(2\beta \omega S)] \} \\ &= \hat{\omega} \frac{\partial}{\partial(\beta \omega)} \ln \left\{ \exp(-\beta \omega S) \frac{1 - \exp[\beta \omega(2S + 1)]}{1 - \exp(\beta \omega)} \right\} \\ &= \hat{\omega} \frac{\partial}{\partial(\beta \omega)} \ln \frac{\exp[-\beta \omega(S + 1/2)] - \exp[\beta \omega(S + 1/2)]}{\exp(-\beta \omega/2) - \exp(\beta \omega/2)} \\ &= \hat{\omega} \frac{\partial}{\partial(\beta \omega)} \ln \frac{\sinh[\beta \omega(S + 1/2)]}{\sinh[\beta \omega/2]}\end{aligned}\tag{11}$$

where $\hat{\omega}$ is the unit vector along the direction of ω .

Spin in non-relativistic thermal systems

At the limit $\omega \ll T$ or $\beta\omega \ll 1$, the spin vector in Eq. (11) becomes:

$$\begin{aligned}\mathbf{S} &= \frac{1}{3}S(S+1)\beta\omega \\ \mathbf{P} &= \frac{1}{3}(S+1)\beta\omega\end{aligned}\tag{12}$$

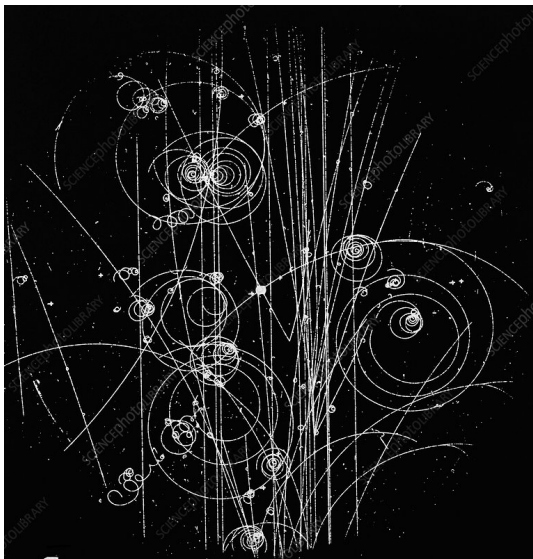
Then we obtain

$$\begin{aligned}\mathbf{S} &= \frac{1}{2}\mathbf{P} \approx \frac{1}{4}\beta\omega, \quad (S = 1/2) \\ \mathbf{S} &= \mathbf{P} \approx \frac{2}{3}\beta\omega, \quad (S = 1) \\ \mathbf{S} &= \frac{3}{2}\mathbf{P} \approx \frac{5}{4}\beta\omega, \quad (S = 3/2)\end{aligned}\tag{13}$$

Spin in relativistic quantum theory

- What is a microscopic particle? How to describe it (mass, spin, charge, parity)?
- Quantum + Special Relativity

Particle tracks in bubble chamber



Translation and Lorentz transformation

Let x^μ be the coordinates of a four-vector in frame F and x'^μ is the coordinates of a four-vector in frame F' , then the translation and Lorentz transformation is expressed as

$$x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu \quad (14)$$

which can be denoted as $\{a, \Lambda\}$ with a^μ being a four vector and Λ a 4×4 matrix satisfying

$$g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = g_{\rho\sigma} \quad (15)$$

Note that the positions of two indices are sensitive in Λ^μ_ν , they can be lowered or raised by the metrix tensor $g_{\mu\nu}$ or $g^{\mu\nu}$. The distance is invariant under Lorentz transformation

$$g_{\mu\nu} dx'^\mu dx'^\nu = g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma dx^\rho dx^\sigma = g_{\rho\sigma} dx^\rho dx^\sigma \quad (16)$$

Lorentz transformation

We have following property for Λ^μ_ν

$$\begin{aligned}\det(g_{\mu\nu}\Lambda^\mu_\rho\Lambda^\nu_\sigma) &= (\det\Lambda)^2 \det g = \det g \\ \Rightarrow \det\Lambda &= \pm 1\end{aligned}\tag{17}$$

We can also derive from (15)

$$\begin{aligned}(g_{\mu\nu}\Lambda^\mu_\rho\Lambda^\nu_\sigma)\Lambda^\kappa_\tau g^{\sigma\tau} &= g_{\rho\sigma}\Lambda^\kappa_\tau g^{\sigma\tau} = \Lambda^\kappa_\rho \\ g_{\mu\nu}\Lambda^\mu_\rho(\Lambda^\nu_\sigma\Lambda^\kappa_\tau g^{\sigma\tau}) &= g_{\mu\nu}\Lambda^\mu_\rho g^{\nu\kappa} = \Lambda^\kappa_\rho \\ &\downarrow \\ \Lambda^\nu_\sigma\Lambda^\kappa_\tau g^{\sigma\tau} &= g^{\nu\kappa}\end{aligned}\tag{18}$$

Lorentz transformation

The inverse Lorentz transformation is derived as

$$\begin{aligned}g_{\mu\nu}\Lambda^\mu_\rho\Lambda^\nu_\sigma &= g_{\rho\sigma} \Rightarrow g_{\mu\nu}\Lambda^\mu_\rho\Lambda^\nu_\sigma g^{\rho\lambda} = g^\lambda_\sigma \\ \Lambda^\lambda_\nu\Lambda^\nu_\sigma &= g^\lambda_\sigma = (\Lambda^{-1})^\lambda_\nu\Lambda^\nu_\sigma \\ &\downarrow \\ (\Lambda^{-1})^\lambda_\nu &= \Lambda^\lambda_\nu = g_{\nu\rho}\Lambda^\rho_\alpha g^{\lambda\alpha}\end{aligned}\tag{19}$$

or in another way

$$\begin{aligned}\Lambda^\lambda_\alpha\Lambda^\beta_\lambda &= g^\beta_\alpha = \Lambda^\lambda_\alpha(\Lambda^{-1})^\beta_\lambda \\ &\downarrow \\ (\Lambda^{-1})^\beta_\lambda &= \Lambda^\beta_\lambda\end{aligned}\tag{20}$$

Lorentz transformation

The Lorentz transformation for x^μ reads

$$x'^\mu = \Lambda^\mu_\nu x^\nu \quad (21)$$

The inverse transformation

$$\begin{aligned} (\Lambda^{-1})^\lambda_\mu x'^\mu &= (\Lambda^{-1})^\lambda_\mu \Lambda^\mu_\nu x^\nu = x^\lambda \\ &\downarrow \\ x^\lambda &= (\Lambda^{-1})^\lambda_\mu x'^\mu = x'^\mu \Lambda_\mu^\lambda \end{aligned} \quad (22)$$

Poincare group (inhomogeneous Lorentz group)

The translation and Lorentz transformation (14) form Poincare group or inhomogeneous group

$$\begin{aligned}x''^\mu &= \Lambda'^\mu_\rho x'^\rho + a'^\mu = \Lambda'^\mu_\rho (\Lambda^\rho_\nu x^\nu + a^\rho) + a'^\mu \\ &= \Lambda'^\mu_\rho \Lambda^\rho_\nu x^\nu + \Lambda'^\mu_\rho a^\rho + a'^\mu\end{aligned}\quad (23)$$

One can check that $\Lambda'^\mu_\rho \Lambda^\rho_\nu$ satisfies (15)

$$\begin{aligned}g_{\mu\alpha} (\Lambda'^\mu_\rho \Lambda^\rho_\nu) (\Lambda'^\alpha_\sigma \Lambda^\sigma_\beta) &= (g_{\mu\alpha} \Lambda'^\mu_\rho \Lambda'^\alpha_\sigma) (\Lambda^\rho_\nu \Lambda^\sigma_\beta) \\ &= g_{\rho\sigma} \Lambda^\rho_\nu \Lambda^\sigma_\beta = g_{\nu\beta}\end{aligned}\quad (24)$$

So $\Lambda'^\mu_\rho \Lambda^\rho_\nu$ is also a Lorentz transformation.

Homogeneous Lorentz group

We can set $a = 0$, this gives the homogeneous Lorentz group

$$U(\Lambda', 0)U(\Lambda, 0) = U(\Lambda'\Lambda, 0) \quad (25)$$

The inverse of the transformation (14) is

$$\begin{aligned} x^\lambda &= (\Lambda^{-1})^\lambda{}_\mu \Lambda^\mu{}_\nu x'^\nu = (\Lambda^{-1})^\lambda{}_\mu (x'^\mu - a^\mu) \\ &= (\Lambda^{-1})^\lambda{}_\mu x'^\mu - (\Lambda^{-1})^\lambda{}_\mu a^\mu \end{aligned} \quad (26)$$

which can be represented by $U(\Lambda^{-1}, -\Lambda^{-1}a)$. We can check it is really the inverse of (14)

$$\begin{aligned} U(\Lambda, a)U(\Lambda^{-1}, -\Lambda^{-1}a) &= U(\Lambda^{-1}, -\Lambda^{-1}a)U(\Lambda, a) \\ &= U(1, 0) \end{aligned} \quad (27)$$

Poincare group (inhomogeneous group)

The transformation (23) can be represented as

$$U(\Lambda', a')U(\Lambda, a) = U(\Lambda'\Lambda, \Lambda'a + a') \quad (28)$$

Form (17), we have $\det\Lambda = \pm 1$. The transformation with $\det\Lambda = +1$ form a subgroup of Lorentz group. From (15) we obtain for $\rho = \sigma = 0$

$$(\Lambda^0_0)^2 = 1 + (\Lambda^i_0)^2 > 1 \quad (29)$$

which lead to the inequalities for Λ^0_0 :

$$\Lambda^0_0 > 1, \quad \Lambda^0_0 < -1 \quad (30)$$

Proper orthochronous Lorentz group

The conditions

$$\det\Lambda = +1 \quad \text{and} \quad \Lambda^0_0 > 1 \quad (31)$$

define the proper orthochronous Lorentz group, a subgroup of Poincare group.

It is impossible to jump from $\det\Lambda = +1$ to $\det\Lambda = -1$ or from $\Lambda^0_0 > 1$ to $\Lambda^0_0 < -1$ by a continuous change of parameters. Any Lorentz transformation that can be obtained from the identity by a continuous change of parameters must have $\det\Lambda = +1$ and $\Lambda^0_0 > 1$, the same sign as the identity, and hence belong to the proper orthochronous Lorentz group.

Proper orthochronous Lorentz group

Any Lorentz transformation is either proper and orthochronous, or may be written as the product of an element of the proper orthochronous Lorentz group with one of the discrete transformations $\mathcal{P}$ or $\mathcal{T}$ or $\mathcal{PT}$, where $\mathcal{P}$ is the space inversion, whose non-zero elements are

$$\mathcal{P}_0^0 = 1, \mathcal{P}_1^1 = \mathcal{P}_2^2 = \mathcal{P}_3^3 = -1 \quad (32)$$

and $\mathcal{T}$ is the time-reversal matrix, whose non-zero elements are

$$\mathcal{T}_0^0 = -1, \mathcal{T}_1^1 = \mathcal{T}_2^2 = \mathcal{T}_3^3 = 1 \quad (33)$$

Thus the study of the whole Lorentz group reduces to the study of its proper orthochronous subgroup, plus space inversion and time-reversal.

Much of the information about any Lie symmetry group is contained in properties of the group elements near the identity, For the inhomogeneous Lorentz group, the identity is the transformation $\Lambda^\mu_\nu = g^\mu_\nu$, $a^\mu = 0$, so we want to study those transformations with

$$\Lambda^\mu_\nu = g^\mu_\nu + \omega^\mu_\nu, \quad a^\mu = \epsilon^\mu \quad (34)$$

where both ω^μ_ν and ϵ^μ are taken infinitesimal. Eq. (15) becomes

$$\begin{aligned} g_{\rho\sigma} &= g_{\mu\nu} (g^\mu_\rho + \omega^\mu_\rho) (g^\nu_\sigma + \omega^\nu_\sigma) \\ &= g_{\sigma\rho} + \omega_{\sigma\rho} + \omega_{\rho\sigma} + O(\omega^2) \\ &\rightarrow \omega_{\sigma\rho} + \omega_{\rho\sigma} = 0, \quad \text{at } O(\omega^2) \end{aligned} \quad (35)$$

We have 10 independent components: $\omega_{\sigma\rho}$ (6) and ϵ^μ (4).

For an infinitesimal Lorentz transformation (34), we have

$$U(1 + \omega, \epsilon) = 1 + \frac{1}{2}i\omega_{\rho\sigma}J^{\rho\sigma} - i\epsilon_{\rho}P^{\rho} + \dots \quad (36)$$

Here $J^{\rho\sigma}$ and P^{ρ} are ω - and ϵ -independent operators, and the dots denote terms of higher order in ω and/or ϵ . In order for $U(1 + \omega, \epsilon)$ to be unitary, the operators $J^{\rho\sigma}$ and P^{ρ} must be Hermitian

$$\begin{aligned} U^{\dagger}(1 + \omega, \epsilon) &= U^{-1}(1 + \omega, \epsilon) \\ &\downarrow \\ J^{\rho\sigma\dagger} &= J^{\rho\sigma}, \quad P^{\rho\dagger} = P^{\rho} \end{aligned} \quad (37)$$

Since $\omega_{\rho\sigma}$ is antisymmetric, we can take its coefficient $J^{\rho\sigma}$ to be antisymmetric also

$$J^{\rho\sigma} = -J^{\sigma\rho} \quad (38)$$

We now examine the Lorentz transformation properties of $J^{\rho\sigma}$ and P^ρ . We consider the product

$$U(\Lambda, a)U(1 + \omega, \epsilon)U^{-1}(\Lambda, a) \quad (39)$$

Here the inverse transformation is

$$U^{-1}(\Lambda, a) = U(\Lambda^{-1}, -\Lambda^{-1}a) \quad (40)$$

Inserting (36) into (39) we obtain

$$\begin{aligned} & U(\Lambda, a)U(1 + \omega, \epsilon)U^{-1}(\Lambda, a) \\ & \approx 1 + \frac{1}{2}i\omega_{\rho\sigma}U(\Lambda, a)J^{\rho\sigma}U^{-1}(\Lambda, a) - i\epsilon_\rho U(\Lambda, a)P^\rho U^{-1}(\Lambda, a) \end{aligned} \quad (41)$$

On the other hand, we can work out three transformation by the rule (28)

$$\begin{aligned}
 & U(\Lambda, a)U(1 + \omega, \epsilon)U^{-1}(\Lambda, a) \\
 &= U(\Lambda(1 + \omega), \Lambda\epsilon + a) U(\Lambda^{-1}, -\Lambda^{-1}a) \\
 &= U(\Lambda(1 + \omega)\Lambda^{-1}, -\Lambda(1 + \omega)\Lambda^{-1}a + \Lambda\epsilon + a) \\
 &= U(1 + \Lambda\omega\Lambda^{-1}, \Lambda\epsilon - \Lambda\omega\Lambda^{-1}a) \\
 &\approx 1 + \frac{1}{2}i(\Lambda\omega\Lambda^{-1})_{\rho\sigma}J^{\rho\sigma} - i(\Lambda\epsilon - \Lambda\omega\Lambda^{-1}a)_{\rho}P^{\rho}
 \end{aligned} \tag{42}$$

To first order in ω and ϵ , equating coefficients of $\omega_{\rho\sigma}$ and ϵ_{ρ} on both sides of Eqs. (41) and (42), we obtain

$$\begin{aligned}
 U(\Lambda, a)J^{\rho\sigma}U^{-1}(\Lambda, a) &= \Lambda_{\mu}^{\rho}\Lambda_{\nu}^{\sigma}(J^{\mu\nu} + P^{\mu}a^{\nu} - P^{\nu}a^{\mu}) \\
 U(\Lambda, a)P^{\rho}U^{-1}(\Lambda, a) &= \Lambda_{\mu}^{\rho}P^{\mu}
 \end{aligned} \tag{43}$$

In (43), we can also express

$$\begin{aligned}\Lambda_{\mu}^{\rho} P^{\mu} &= (\Lambda^{-1})^{\rho}_{\mu} P^{\mu} \\ \Lambda_{\mu}^{\rho} \Lambda_{\nu}^{\sigma} J^{\mu\nu} &= (\Lambda^{-1})^{\rho}_{\mu} (\Lambda^{-1})^{\sigma}_{\nu} J^{\mu\nu}\end{aligned}\tag{44}$$

Poincare algebra

In (43) we consider infinitesimal transformation $\Lambda^\mu_\nu = g^\mu_\nu + \omega^\mu_\nu$ and $a^\mu = \epsilon^\mu$ and keep only terms of first order in ω^μ_ν and ϵ^μ , Eq. (43) now becomes

$$\begin{aligned} i \left[\frac{1}{2} \omega_{\mu\nu} J^{\mu\nu} - \epsilon_\mu P^\mu, J^{\rho\sigma} \right] &= \omega_\mu^\rho J^{\mu\sigma} + \omega_\nu^\rho J^{\rho\nu} - \epsilon^\rho P^\sigma + \epsilon^\sigma P^\rho \\ i \left[\frac{1}{2} \omega_{\mu\nu} J^{\mu\nu} - \epsilon_\mu P^\mu, P^\rho \right] &= \omega_\mu^\rho P^\mu \end{aligned} \quad (45)$$

Equating coefficients of $\omega_{\mu\nu}$ and ϵ_μ on both sides of these equations, we find the commutation rules for the Lie algebra of Poincare group (or Poincare algebra)

$$\begin{aligned} [P^\mu, P^\nu] &= 0 \\ [P^\mu, J^{\nu\lambda}] &= -i (g^{\lambda\mu} P^\nu - g^{\mu\nu} P^\lambda) \\ [J^{\mu\nu}, J^{\rho\sigma}] &= -i (g^{\mu\rho} J^{\nu\sigma} + g^{\nu\sigma} J^{\mu\rho} - g^{\mu\sigma} J^{\nu\rho} - g^{\nu\rho} J^{\mu\sigma}) \end{aligned} \quad (46)$$

Homework 1: Poincare algebra

Apply infinitesimal transformation $\Lambda^\mu_\nu = g^\mu_\nu + \omega^\mu_\nu$ and $a^\mu = \epsilon^\mu$ in (43), try to prove (45).

Homework 2: Poincare algebra

The differential operator forms for $J^{\mu\nu}$ and P^μ are

$$\begin{aligned}P^\mu &= i\partial^\mu = (i\partial_t, -i\nabla) \\J^{\mu\nu} &= x^\mu P^\nu - x^\nu P^\mu = i(x^\mu \partial^\nu - x^\nu \partial^\mu)\end{aligned}\tag{47}$$

With the commutation rule $[x^\mu, P^\nu] = -ig^{\mu\nu}$ and $[x_\mu, P_\nu] = -ig_{\mu\nu}$, we can prove the Poincare algebra (46).

Lorentz boost and rotation

We can define two groups of algebra

$$\begin{aligned}\mathbf{J} &= (J_{31}, J_{23}, J_{12}) = (J^{31}, J^{23}, J^{12}) \\ \mathbf{K} &= (J^{01}, J^{02}, J^{03}) = -(J_{01}, J_{02}, J_{03})\end{aligned}\quad (48)$$

which corresponds to rotation ($\mathbf{J}$) and Lorentz boost ($\mathbf{K}$). According to (46), they satisfy

$$\begin{aligned}[J_i, J_j] &= i\epsilon_{ijk} J_k \\ [J_i, K_j] &= i\epsilon_{ijk} K_k \\ [K_i, K_j] &= -i\epsilon_{ijk} J_k\end{aligned}\quad (49)$$

We can define two sets of operators

$$\begin{aligned}A_i &= \frac{1}{2}(J_i + iK_i) \\ B_i &= \frac{1}{2}(J_i - iK_i)\end{aligned}\tag{50}$$

They satisfy

$$\begin{aligned}[A_i, A_j] &= i\epsilon_{ijk}A_k \\ [B_i, B_j] &= i\epsilon_{ijk}B_k \\ [A_i, B_j] &= 0\end{aligned}\tag{51}$$

Homework 3: rotation and Lorentz boost

Prove Eqs. (49) and (51) using Eq. (46).

- The SL(2,C) group is the special linear group of 2×2 complex matrices with determinant 1. It is a 6-dimensional Lie group (since 2×2 complex matrices have 4 real parameters, and $\det=1$ imposes one constraint, leaving 6 real parameters). It serves as the universal cover of the proper orthochronous Lorentz group $SO^+(1, 3)$.
- The SL(2,C) group applies to spinor fields. It is crucial for formulating relativistic quantum mechanics and quantum field theory for spin-1/2 particles, as it correctly describes how their spin states transform under Lorentz boosts and rotations.

The group elements are written as

$$\begin{aligned} U(\Lambda) &= e^{-i\mathbf{J}\cdot\boldsymbol{\theta} - i\mathbf{K}\cdot\boldsymbol{\eta}} = e^{-i\mathbf{A}\cdot\boldsymbol{\theta}_A - i\mathbf{B}\cdot\boldsymbol{\theta}_B} \\ &= \exp \left[-i\frac{1}{2}(\mathbf{J} + i\mathbf{K}) \cdot \boldsymbol{\theta}_A - i\frac{1}{2}(\mathbf{J} - i\mathbf{K}) \cdot \boldsymbol{\theta}_B \right] \\ &= \exp \left[-i\frac{1}{2}\mathbf{J} \cdot (\boldsymbol{\theta}_A + \boldsymbol{\theta}_B) + \frac{1}{2}\mathbf{K} \cdot (\boldsymbol{\theta}_A - \boldsymbol{\theta}_B) \right] \end{aligned} \quad (52)$$

where

$$\boldsymbol{\theta}_A = \boldsymbol{\theta} - i\boldsymbol{\eta}, \quad \boldsymbol{\theta}_B = \boldsymbol{\theta} + i\boldsymbol{\eta} \quad (53)$$

One can verify $U^\dagger(\Lambda) = U^{-1}(\Lambda)$, i.e. $U(\Lambda)$ is unitary.

For the irreducible $(S, 0)$ and $(0, S)$ representations (S is the particle's spin quantum number), we have

$$\begin{aligned} D^{(S)}(\Lambda) &= e^{-i\mathbf{A}\cdot\boldsymbol{\theta}_A} = \exp [(-i\theta - \boldsymbol{\eta}) \cdot \mathbf{A}] \\ \overline{D}^{(S)}(\Lambda) &= e^{-i\mathbf{B}\cdot\boldsymbol{\theta}_B} = \exp [(-i\theta + \boldsymbol{\eta}) \cdot \mathbf{B}] \end{aligned} \quad (54)$$

They are related to each other

$$D^{(S)\dagger}(\Lambda^{-1}) = \overline{D}^{(S)}(\Lambda) \quad (55)$$

Note that $D^{(S)}(\Lambda)$ and $D^{(S)\dagger}(\Lambda^{-1})$ are two inequivalent finite-dimensional non-unitary representations of the Lorentz group and correspond to those labeled as $(S, 0)$ and $(0, S)$ respectively.

As usual we shall look for a maximal set of commuting observables. The four energy-momentum operators P^μ commute. Let p^μ be an “eigenvalue” of P^μ . Let us now look for Lorentz transformations which leave p^μ invariant, and write

$$U(\Lambda) = \exp \left(\frac{i}{2} \omega_{\mu\nu} \mathcal{J}^{\mu\nu} \right) \quad (56)$$

where $\mathcal{J}^{\mu\nu}$ are 4×4 matrices as a particular representation for the generators $J^{\mu\nu}$

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} = -i \left(g_\alpha^\mu g_\beta^\nu - g_\beta^\mu g_\alpha^\nu \right) \quad (57)$$

One can check that the above satisfies the commutation rules in (46)

Pauli-Lubanski pseudo-vector

The invariance of p implies $\omega_{\mu\nu}p^\nu$ which is solved according to $\omega_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}s^\rho p^\sigma$ where $\epsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor $\epsilon_{0123} = -\epsilon^{0123} = -1$, and s^μ a four-vector whose component along p^μ is irrelevant. One can check

$$\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})_{\alpha\beta}p^\beta = -i\omega_{\mu\nu}(g_\alpha^\mu p^\nu - g_\alpha^\nu p^\mu) = 0 \quad (58)$$

Thus the Lorentz transformations which leave p^μ invariant are represented as

$$U(\Lambda(p)) = \exp\left(\frac{i}{2}\epsilon_{\rho\mu\nu\sigma}\mathcal{J}^{\mu\nu}p^\sigma s^\rho\right) \quad (59)$$

Pauli-Lubanski pseudo-vector

Here we defined Pauli-Lubanski pseudo-vector

$$W_\rho = -\frac{1}{2}\epsilon_{\rho\mu\nu\sigma}J^{\mu\nu}P^\sigma = -\frac{1}{2}\epsilon_{\rho\mu\nu\sigma}P^\sigma J^{\mu\nu} \quad (60)$$

Due to $\epsilon_{\rho\mu\nu\sigma}[P^\sigma, J^{\mu\nu}] = 0$ from (46), the position of two operators $J^{\mu\nu}$ and P^σ does not matter. We have the identity

$$W(P) \cdot P = 0 \quad (61)$$

One easily finds from (46)

$$\begin{aligned} [W_\mu, P_\nu] &= 0, \quad [W_\mu, W_\nu] = -i\epsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma \\ [J_{\mu\nu}, W_\rho] &= i(g_{\nu\rho}W_\mu - g_{\mu\rho}W_\nu) \end{aligned} \quad (62)$$

The latter indicating that W_μ behaves as a four-vector under Lorentz transformations.

Let then $|p, \sigma\rangle$ be a vector of representation space such that $P^\mu |p, \sigma\rangle = p^\mu$, so we have

$$U(\Lambda(p)) |p, \sigma\rangle = \exp(-iW_\rho s^\rho) |p, \sigma\rangle \quad (63)$$

It can be shown that

$$\begin{aligned} P^\mu \exp(-iW_\rho s^\rho) |p, \sigma\rangle &= \exp(-iW_\rho s^\rho) P^\mu |p, \sigma\rangle \\ &= p^\mu \exp(-iW_\rho s^\rho) |p, \sigma\rangle \end{aligned} \quad (64)$$

because $[W_\mu, P_\nu] = 0$.

Pauli-Lubanski pseudo-vector

From these commutation rules one observes that $P_\mu P^\mu = P^2$ and $W_\mu W^\mu = W^2$ commute with all P^μ and $J^{\mu\nu}$, so $P^2 = m^2$ and W^2 are two commutable invariant operators. The proof is simple

$$\begin{aligned}[J_{\mu\nu}, g^{\rho\sigma} W_\rho W_\sigma] &= ig^{\rho\sigma} (g_{\nu\rho} W_\mu - g_{\mu\rho} W_\nu) W_\sigma + ig^{\rho\sigma} W_\rho (g_{\nu\sigma} W_\mu - g_{\mu\sigma} W_\nu) \\ &= i(W_\nu W_\mu - W_\mu W_\nu) + i(W_\mu W_\nu - W_\nu W_\mu) = 0 \\ [J_{\mu\nu}, g^{\rho\sigma} P_\rho P_\sigma] &= ig^{\rho\sigma} (g_{\nu\rho} P_\mu - g_{\mu\rho} P_\nu) P_\sigma + ig^{\rho\sigma} P_\rho (g_{\nu\sigma} P_\mu - g_{\mu\sigma} P_\nu) \\ &= i(P_\mu P_\nu - P_\nu P_\mu) + i(P_\nu P_\mu - P_\mu P_\nu) = 0\end{aligned}\quad (65)$$

So $P^2 = m^2$ and $W^2/m^2 = S(S+1)$ (spin quantum number) can be regarded as two intrinsic constants to characterize a particle.

One can attach to each p three space-like vectors $n^{(i)\mu}(p)$ ($i = 1, 2, 3$)

$$\begin{aligned}n^{(i)}(p) \cdot n^{(j)}(p) &= -\delta_{ij} \\p \cdot n^{(i)}(p) &= 0 \\g_{ij} n_{\mu}^{(i)}(p) n_{\nu}^{(j)}(p) &= g_{\mu\nu} \\\epsilon^{\mu\nu\rho\sigma} n_{\mu}^{(\alpha)} n_{\nu}^{(\beta)} n_{\rho}^{(\gamma)} n_{\sigma}^{(\delta)} &= -\epsilon^{\alpha\beta\gamma\delta}\end{aligned}\tag{66}$$

where we assume $n^{(0)\mu}(p) = p^{\mu}/m$. We can expand $W^{\mu}(p)$ as

$$W^{\mu}(p) = \sum_{i=1}^3 W_i(p) n^{(i)\mu}(p)\tag{67}$$

Here we can extract $W_i(p)$ as

$$\begin{aligned} W_i(p) &= -W(p) \cdot n^{(i)}(p) \\ &= \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} J^{\mu\nu} p^\sigma n^{(i)\rho}(p) \end{aligned} \quad (68)$$

So one can define three spin operators

$$S_i(p) = \frac{1}{m} W_i(p) = \frac{1}{2m} \epsilon_{\mu\nu\rho\sigma} J^{\mu\nu} p^\sigma n^{(i)\rho}(p) \quad (69)$$

One can verify the commutators

$$[S_i(p), S_j(p)] = i\epsilon_{ijk} S_k(p) \quad (70)$$

Here $S_i(p)$ are generators of $SU(2)$ subgroup of proper orthochronous Lorentz group which leaves p invariant. Thus we have

$$\sum_{i=1}^3 S_i^2(p) = \frac{1}{m^2} \sum_{i=1}^3 W_i^2(p) = -\frac{1}{m^2} W^2(p) \quad (71)$$

which has possible eigenvalues $S(S+1)$ with S being integer or half-integer. So $P^2 = m^2$ and $\hat{S}^2 = S(S+1)$ can characterize a particle.

One particle spin-momentum state

A particle is defined as a spin-momentum quantum state in its rest frame

$$|m, \sigma\rangle \quad (72)$$

where σ is the spin quantum number along the spin quantization direction (magnetic quantum number)

$$\sigma = -S, -S + 1, -S + 2, \dots, S - 1, S \quad (73)$$

Here S is the spin quantum number which is defined through

$$\begin{aligned} \hat{S}^2 |m, \sigma\rangle &= S(S + 1) |m, \sigma\rangle \\ \hat{S}_z |m, \sigma\rangle &= \sigma |m, \sigma\rangle \end{aligned} \quad (74)$$

where we choose the z-axis as the spin quantization direction.

One particle spin-momentum state

Here m is the particle's mass defined through

$$P^\mu |m, \sigma\rangle = p^\mu |m, \sigma\rangle = (m, 0) |m, \sigma\rangle$$

$$P^2 |m, \sigma\rangle = p^2 |m, \sigma\rangle = m^2 |m, \sigma\rangle$$

where $p^\mu = (m, 0)$ is the four-momentum of the particle at rest.

One particle spin-momentum state

When the particle is not at rest but has an on-shell four-momentum $p^\mu = (E_p, \mathbf{p})$, its spin-momentum state can be represented by $U(L(p))$ as

$$|p, \sigma\rangle \equiv |\mathbf{p}, \sigma\rangle = U(L(p)) |m, \sigma\rangle \quad (75)$$

where $L(p) \equiv L^\mu_\nu(p)$ denotes the Lorentz transformation that transforms a particle at rest to $p^\mu = (E_p, \mathbf{p})$,

$$\begin{aligned} L^0_0 &= \frac{E_p}{m}, \quad L^0_i = \frac{1}{m} \mathbf{p}_i, \quad L^i_0 = \frac{1}{m} \mathbf{p}_i \\ L^i_j &= \delta_{ij} - \hat{\mathbf{p}}_i \hat{\mathbf{p}}_j + \frac{E_p}{m} \hat{\mathbf{p}}_i \hat{\mathbf{p}}_j \end{aligned} \quad (76)$$

The state is normalized as

$$\langle p', \sigma' | p, \sigma \rangle = 2E_p (2\pi)^3 \delta^{(3)}(\mathbf{p}' - \mathbf{p}) \delta_{\sigma' \sigma} \quad (77)$$

Lorentz transformation of one particle state

The Lorentz transformation of a spin-momentum state is

$$\begin{aligned} U(\Lambda) |p, \sigma\rangle &= U(\Lambda) U(L(p)) |m, \sigma\rangle \\ &= U(L(\Lambda p)) U(L^{-1}(\Lambda p) \Lambda L(p)) |m, \sigma\rangle \\ &= \sum_{\sigma'} U(L(\Lambda p)) |m, \sigma'\rangle \langle m, \sigma' | U(L^{-1}(\Lambda p) \Lambda L(p)) |m, \sigma\rangle \\ &= \sum_{\sigma'} |\Lambda p, \sigma'\rangle \langle m, \sigma' | U[L^{-1}(\Lambda p) \Lambda L(p)] |m, \sigma\rangle \\ &= \sum_{\sigma'} |\Lambda p, \sigma'\rangle D_{\sigma'\sigma}^{(S)} [L^{-1}(\Lambda p) \Lambda L(p)] \end{aligned} \quad (78)$$

where we defined D-matrix element for the Wigner rotation R

$$D_{\sigma'\sigma}^{(S)}(R) = \langle m, \sigma' | U(R) |m, \sigma\rangle, \quad R = L^{-1}(\Lambda p) \Lambda L(p) \quad (79)$$

The representation of the little group (Wigner rotation) must be unitary, so we have

$$D^{(S)\dagger}(R) = D^{(S)-1}(R) = D^{(S)}(R^{-1}) \quad (80)$$

Lorentz transformation of one particle state

The creation and annihilation operators transform as

$$U(\Lambda) a_{p,\sigma}^\dagger U^{-1}(\Lambda) = a_{\Lambda p,\sigma'}^\dagger D_{\sigma'\sigma}^{(S)} [L^{-1}(\Lambda p) \Lambda L(p)] \quad (81)$$

$$U(\Lambda) a_{p,\sigma} U^{-1}(\Lambda) = D_{\sigma\sigma'}^{(S)} [L^{-1}(p) \Lambda^{-1} L(\Lambda p)] a_{\Lambda p,\sigma'} \quad (82)$$

The first equality follows Eq. (78) as

$$\begin{aligned} U(\Lambda) |p, \sigma\rangle &= U(\Lambda) a_{p,\sigma}^\dagger U^{-1}(\Lambda) U(\Lambda) |0\rangle \\ &= U(\Lambda) a_{p,\sigma}^\dagger U^{-1}(\Lambda) |0\rangle \\ &= a_{\Lambda p,\sigma'}^\dagger |0\rangle D_{\sigma'\sigma}^{(S)} [L^{-1}(\Lambda p) \Lambda L(p)] \end{aligned} \quad (83)$$

where we used $U(\Lambda) |0\rangle = |0\rangle$ and repetition of indices means summation.

Homework 4: Lorentz transformation of one particle state

Try to derive (82) by taking Hermitian conjugate of (81), using the property that $U(\Lambda)$ is unitary.

Charge conjugate form

For a rotation, we have the relation

$$D^{(S)*}(R) = CD^{(S)}(R)C^{-1} \quad (84)$$

where C is a $(2S + 1) \times (2S + 1)$ unitary matrix satisfying

$$\begin{aligned} C^\dagger &= C^T = C^{-1} = (-1)^{2S} C \\ C^2 &= (-1)^{2S} I \\ C &= C^{-1} (-1)^{2S} \end{aligned} \quad (85)$$

The explicit form of Eq. (84) can be written as

$$D_{\sigma\sigma'}^{(S)}(R) = \left[CD^{(S)}(R^{-1}) C^{-1} \right]_{\sigma'\sigma} \quad (86)$$

Charge conjugate form

By (86) we can rewrite Eqs. (81,82) as

$$\begin{aligned} U(\Lambda) a_{p,\sigma}^\dagger U^{-1}(\Lambda) &= \left(CD^{(S)} [L^{-1}(p)\Lambda^{-1}L(\Lambda p)] C^{-1} \right)_{\sigma\sigma'} a_{\Lambda p,\sigma'}^\dagger \\ U(\Lambda) a_{p,\sigma} U^{-1}(\Lambda) &= a_{\Lambda p,\sigma'} \left(CD^{(S)} [L^{-1}(\Lambda p)\Lambda L(p)] C^{-1} \right)_{\sigma'\sigma} \end{aligned} \quad (87)$$

We can use above equation to define the transformation of the anti-particle operator $b^\dagger$ as

$$\begin{aligned} U(\Lambda) b_{p,\sigma}^\dagger U^{-1}(\Lambda) &= \left(CD^{(S)} [L^{-1}(p)\Lambda^{-1}L(\Lambda p)] C^{-1} \right)_{\sigma\sigma'} b_{\Lambda p,\sigma'}^\dagger \\ U(\Lambda) b_{p,\sigma} U^{-1}(\Lambda) &= b_{\Lambda p,\sigma'} \left(CD^{(S)} [L^{-1}(\Lambda p)\Lambda L(p)] C^{-1} \right)_{\sigma'\sigma} \end{aligned} \quad (88)$$

Lorentz transformation of quantum fields

We use the symbols

$$\begin{aligned}\alpha_{p,\sigma} &= D_{\sigma\sigma'}(L(p)) a_{p,\sigma'} \\ \beta_{p,\sigma} &= [D(L(p)) C^{-1}]_{\sigma\sigma'} b_{p,\sigma'}^\dagger \\ \bar{\alpha}_{p,\sigma} &= D_{\sigma\sigma'}^{(S)\dagger}(L^{-1}(p)) a_{p,\sigma'} \\ \bar{\beta}_{p,\sigma} &= [D^{(S)\dagger}(L^{-1}(p)) C]_{\sigma\sigma'} b_{p,\sigma'}^\dagger\end{aligned}\tag{89}$$

We can verify

$$U(\Lambda)O(p,\sigma)U^{-1}(\Lambda) = D_{\sigma\sigma'}(\Lambda^{-1}) O(\Lambda p, \sigma')\tag{90}$$

where $O(p,\sigma) = \alpha_{p,\sigma}, \beta_{p,\sigma}, \bar{\alpha}_{p,\sigma}, \bar{\beta}_{p,\sigma}$.

Homework 5: Lorentz transformation of quantum fields

Prove Eq. (90).

Lorentz transformation of quantum fields

So we can construct the field with $\alpha_{p,\sigma}$, $\beta_{p,\sigma}$, $\bar{\alpha}_{p,\sigma}$, $\bar{\beta}_{p,\sigma}$

$$\begin{aligned}\psi_\sigma(x) &= \int \frac{d^3p}{(2\pi)^3 2E_p} \alpha_{p,\sigma} e^{-ipx} + \beta_{p,\sigma} e^{ipx} \\ \tilde{\psi}_\sigma(x) &= \int \frac{d^3p}{(2\pi)^3 2E_p} \bar{\alpha}_{p,\sigma} e^{-ipx} + \bar{\beta}_{p,\sigma} e^{ipx}\end{aligned}\tag{91}$$

By (90) we can prove

$$\begin{aligned}U(\Lambda) \psi_\sigma(x) U^{-1}(\Lambda) &= D_{\sigma\tau}^{(S)}(\Lambda^{-1}) \psi_\tau(\Lambda x) \\ U(\Lambda) \tilde{\psi}_\sigma(x) U^{-1}(\Lambda) &= D_{\sigma\tau}^{(S)}(\Lambda^{-1}) \tilde{\psi}_\tau(\Lambda x)\end{aligned}\tag{92}$$

Lorentz transformation of quantum fields

The (anti-) commutation relations of creation and annihilation operators in (91) are related to the normalization of the states (77) and read

$$\begin{aligned}\left[a_{\mathbf{p},\sigma}, a_{\mathbf{p}',\sigma'}^\dagger\right]_{\pm} &= \left[b_{\mathbf{p},\sigma}, b_{\mathbf{p}',\sigma'}^\dagger\right]_{\pm} \\ &= 2E_{\mathbf{p}}(2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}') \delta_{\sigma'\sigma}\end{aligned}\quad (93)$$

The $2(2S + 1)$ field degrees of freedom in (91) are needed to represent particles and antiparticles. We have

$$\tilde{\psi}_\tau(x) = C_{\tau\sigma} \psi_\sigma^{c\dagger}(x) \quad (94)$$

where $\psi_\sigma^c(x)$ is the charge-conjugated field, which is obtained from $\psi_\sigma(x)$ by swapping $a \leftrightarrow b$ and $a^\dagger \leftrightarrow b^\dagger$ in (91).

Homework 6: Lorentz transformation of quantum fields

Prove Eq. (94).

Quantum fields for spin- S particles

We can define a $2(2S + 1)$ -component field

$$\begin{aligned}\Psi(x) &= \int \frac{d^3p}{(2\pi)^3 2E_p} \left[U(p) a_p e^{-ip \cdot x} + V(p) b_p^\dagger e^{ip \cdot x} \right] \\ \bar{\Psi}(x) &= \int \frac{d^3p}{(2\pi)^3 2E_p} \left[a_p^\dagger \bar{U}(p) e^{ip \cdot x} + b_p \bar{V}(p) e^{-ip \cdot x} \right]\end{aligned}\quad (95)$$

where a_p and $b_p^\dagger$ are $(2S + 1)$ -dimensional column vectors of the annihilation and creation operators., and $2(2S + 1)$ -component spinors

$$\begin{aligned}U(p) &= \begin{pmatrix} D^{(S)}(L(p)) \\ D^{(S)\dagger}(L^{-1}(p)) \end{pmatrix}, \quad V(p) = \begin{pmatrix} D^{(S)}(L(p)) C^{-1} \\ D^{(S)\dagger}(L^{-1}(p)) C \end{pmatrix} \\ \bar{U}(p) &= U^\dagger(p) \Gamma_0 = \left[D^{(S)}(L^{-1}(p)), D^{(S)\dagger}(L(p)) \right] \\ \bar{V}(p) &= V^\dagger(p) \Gamma_0 = \left[C^{-1} D^{(S)}(L^{-1}(p)), C D^{(S)\dagger}(L(p)) \right]\end{aligned}\quad (96)$$

Quantum fields for spin- S particles

Here Γ_0 is the $2(2S+1) \times 2(2S+1)$ matrix

$$\Gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad (97)$$

Here are some properties of spinors

$$\begin{aligned} \bar{U}(p)U(p) &= \left[D^{(S)}(L^{-1}(p)), D^{(S)\dagger}(L(p)) \right] \begin{pmatrix} D^{(S)}(L(p)) \\ D^{(S)\dagger}(L^{-1}(p)) \end{pmatrix} = 2I \\ \bar{V}(p)V(p) &= \left[C^{-1}D^{(S)}(L^{-1}(p)), CD^{(S)\dagger}(L(p)) \right] \begin{pmatrix} D^{(S)}(L(p))C^{-1} \\ D^{(S)\dagger}(L^{-1}(p))C \end{pmatrix} \\ &= 2C^2 = 2(-1)^{2S} \end{aligned} \quad (98)$$

Quantum fields for spin- S particles

$$\begin{aligned}
 U(p)\bar{U}(p) &= \begin{pmatrix} D^{(S)}(L(p)) \\ D^{(S)\dagger}(L^{-1}(p)) \end{pmatrix} \begin{pmatrix} D^{(S)}(L^{-1}(p)), D^{(S)\dagger}(L(p)) \end{pmatrix} \\
 &= \begin{pmatrix} I & D^{(S)}[L(p)L^\dagger(p)] \\ D^{(S)}[(L(p)L^\dagger(p))^{-1}] & I \end{pmatrix} \\
 V(p)\bar{V}(p) &= \begin{pmatrix} (-1)^{2S}I & D^{(S)}[L(p)L^\dagger(p)] \\ D^{(S)}[(L(p)L^\dagger(p))^{-1}] & (-1)^{2S}I \end{pmatrix} \quad (99)
 \end{aligned}$$

The spin-momentum states can be extracted as

$$\begin{aligned}
 |p, \sigma\rangle &= a_{p,\sigma}^\dagger |0\rangle = E_p \int d^4x \bar{\Psi}_\tau(x) U_{\tau\sigma}(p) e^{-ip \cdot x} |0\rangle \\
 \langle p, \sigma| &= \langle 0| a_{p,\sigma} = E_p \langle 0| \int d^4x \bar{U}_{\sigma\tau}(p) \Psi_\tau(x) e^{ip \cdot x} \quad (100)
 \end{aligned}$$

Density operator

The density operator in a rotation frame is defined as

$$\hat{\rho} = \frac{1}{Z} \exp \left[\beta \left(-\hat{H} + \mu \hat{Q} + \omega \cdot \hat{\mathbf{S}} \right) \right] P_V \quad (101)$$

where P_V is the projection operator onto the localized states $|h_V\rangle$, where V means the field exists in the volume V and vanishes out of the volume. The one-particle matrix elements of the density operator are

$$\begin{aligned} f_{\tau\sigma}(p) &\sim \langle p, \tau | \exp \left[\beta \left(-\hat{H} + \mu \hat{Q} + \omega \cdot \hat{\mathbf{J}} \right) \right] P_V | p, \sigma \rangle \\ &= e^{-\beta E_p + \beta \mu q} \langle p, \tau | \exp \left(\beta \omega \cdot \hat{\mathbf{J}} \right) P_V | p, \sigma \rangle \\ &= e^{-\beta E_p + \beta \mu q} \langle p, \tau | \hat{R}_\omega(\phi) P_V | p, \sigma \rangle \\ &= e^{-\beta E_p + \beta \mu q} \int \frac{d^3 p'}{(2\pi)^3 2E_{p'}} \langle p, \tau | \hat{R}_\omega(\phi) | p', \sigma' \rangle \\ &\quad \times \langle p', \sigma' | P_V | p, \sigma \rangle \end{aligned} \quad (102)$$

Here $\hat{R}_\omega(\phi)$ is the rotation around ω of an angle $\phi = i\beta\omega$. Inserting a compact basis into the $\hat{R}$ and P_V , we have

$$\begin{aligned}
 \langle p, \tau | \hat{R}_\omega(\phi) | p', \sigma' \rangle &= \langle p, \tau | R_\omega p', \sigma \rangle D_{\sigma\sigma'}^{(S)} \left[L^{-1} \left(\hat{R}_\omega p' \right) \hat{R}_\omega L(p') \right] \\
 &= 2E_p (2\pi)^3 \delta^{(3)}(p - \hat{R}_\omega p') \delta_{\tau\sigma} \\
 &\quad \times D_{\sigma\sigma'}^{(S)} \left[L^{-1} \left(\hat{R}_\omega p' \right) \hat{R}_\omega L(p') \right] \\
 &= 2E_p (2\pi)^3 \delta^{(3)}(p - \hat{R}_\omega p') \\
 &\quad \times D_{\tau\sigma'}^{(S)} \left[L^{-1} \left(\hat{R}_\omega p' \right) \hat{R}_\omega L(p') \right] \quad (103)
 \end{aligned}$$

Here the $D^{(S)}$ is the representation of $SL(2, \mathbb{C})$ with spin S .

Matrix elements of density operators

For the matrix element of P_V we have

$$\begin{aligned} \langle p', \sigma' | P_V | p, \sigma \rangle &= \sum_{h_V} \langle p', \sigma' | h_V \rangle \langle h_V | p, \sigma \rangle \\ &= E_p E_{p'} \int dx_1 dx_2 \langle 0 | \bar{U}_{\sigma' \tau_1} \Psi_{\tau_1}(x_1) e^{ip' \cdot x_1} | h_V \rangle \\ &\quad \times \langle h_V | \bar{\Psi}_{\tau_2}(x_2) U_{\tau_2 \sigma} e^{-ip \cdot x_2} | 0 \rangle \\ &= E_p E_{p'} \int dx_1 dx_2 e^{ip' \cdot x_1} e^{-ip \cdot x_2} \bar{U}_{\sigma' \tau_1}(p') U_{\tau_2 \sigma}(p) \\ &\quad \times \sum_{h_V} \langle 0 | \Psi_{\tau_1}(x_1) | h_V \rangle \langle h_V | \bar{\Psi}_{\tau_2}(x_2) | 0 \rangle \\ &= E_p E_{p'} \int dx_1 dx_2 e^{ip' \cdot x_1} e^{-ip \cdot x_2} \langle 0 | h_V \rangle \langle h_V | 0 \rangle \\ &\quad \times \bar{U}_{\sigma' \tau_1}(p') U_{\tau_2 \sigma}(p) \sum_{h_V} \Psi_{V, \tau_1}(x_1) \bar{\Psi}_{V, \tau_2}(x_2) \end{aligned} \quad (104)$$

Matrix elements of density operators

Here $|h_V\rangle$ is the eigenstate of $\Psi(x)$ with the eigenvalue Ψ_V inside the volume V

$$\psi_V(x) = \begin{cases} 0 & \mathbf{x} \notin V \\ \Psi(\mathbf{x}) & \mathbf{x} \in V \end{cases} \quad (105)$$

We make an ansatz

$$\begin{aligned} W_{\tau_1\tau_2} \left(x + \frac{y}{2}, x - \frac{y}{2} \right) &= \sum_{h_V} \psi_{V,\tau_1}(x_1) \bar{\psi}_{V,\tau_2}(x_2) \langle 0|h_V\rangle \langle h_V|0\rangle \\ &= \delta_{\tau_1\tau_2} W \left(x + \frac{y}{2}, x - \frac{y}{2} \right) \end{aligned} \quad (106)$$

where we used $x_{1,2} = x \pm y/2$.

Matrix elements of density operators

Then we obtain

$$\begin{aligned}\text{Int} &= \int dx_1 dx_2 e^{ip' \cdot (x+y/2)} e^{-ip \cdot (x-y/2)} W_{\tau_1 \tau_2} \left(x + \frac{y}{2}, x - \frac{y}{2} \right) \\ &= \delta_{\tau_1 \tau_2} \int dx dy e^{-i(p-p') \cdot x} e^{i(p+p') \cdot y/2} W \left(x + \frac{y}{2}, x - \frac{y}{2} \right) \\ &= \delta_{\tau_1 \tau_2} \widetilde{W}(p-p', p+p')\end{aligned}\quad (107)$$

With above relation, Eq. (104) becomes

$$\langle p', \sigma' | P_V | p, \sigma \rangle = E_p E_{p'} \widetilde{W}(p-p', p+p') [\overline{U}(p') U(p)]_{\sigma' \sigma} \quad (108)$$

Matrix elements of density operators

We put (103) and (104) into ()

$$\begin{aligned}
 f_{\tau\sigma}(p) &\sim \left\langle p, \tau \left| \exp \left[\beta \left(-\hat{h} + \mu \hat{q} + \omega \cdot \hat{\mathbf{J}} \right) \right] P_V \right| p, \sigma \right\rangle \\
 &= e^{-\beta E_p + \beta \mu q} \int \frac{d^3 p'}{(2\pi)^3 2E_{p'}} \left\langle p, \tau \left| \hat{R}_\omega(\phi) \right| p', \sigma' \right\rangle \left\langle p', \sigma' \left| P_V \right| p, \sigma \right\rangle \\
 &= E_p^2 e^{-\beta E_p + \beta \mu q} \int d^3 p' \delta^{(3)}(p - \hat{R}_\omega p') D_{\tau\sigma'}^{(S)} \left[L^{-1}(\hat{R}_\omega p') \hat{R}_\omega L(p') \right] \\
 &\quad \times \widetilde{W}(p - p', p + p') \\
 &\quad \times \left[D^{(S)}(L^{-1}(p') L(p)) + D^{(S)}(L^{-1}(p) L(p'))^\dagger \right]_{\sigma'\sigma} \\
 &\approx E_p^2 e^{-\beta E_p + \beta \mu q} \widetilde{W}(p - \hat{R}_\omega^{-1} p, p + \hat{R}_\omega^{-1} p) \\
 &\quad \times \left\{ D_{\tau\sigma}^{(S)} \left[L^{-1}(p) \hat{R}_\omega L(p) \right] + D_{\tau\sigma}^{(S)} \left[L^\dagger(p) \hat{R}_\omega L^{-1\dagger}(p) \right] \right\}
 \end{aligned}$$

So the distribution can be defined as

$$f_{\tau\sigma}(p) = \frac{1}{2} e^{-\beta E_p + \beta \mu q} \times \left\{ D_{\tau\sigma}^{(S)} \left[L^{-1}(p) \hat{R}_\omega L(p) \right] + D_{\tau\sigma}^{(S)} \left[L^\dagger(p) \hat{R}_\omega L^{-1\dagger}(p) \right] \right\} \quad (109)$$

Taking the trace of the density matrix,

$$f(p) = e^{-\beta E_p + \beta \mu q} \text{Tr} \left[D^{(S)}(i\beta\omega) \right] = e^{-\beta E_p + \beta \mu q} \chi(\beta\omega) \quad (110)$$

where

$$\chi(\beta\omega) = \text{Tr} \left[D^{(S)}(i\beta\omega) \right] = \frac{\sinh \left[\beta\omega \left(S + \frac{1}{2} \right) \right]}{\sinh \left(\frac{1}{2} \beta\omega \right)} \quad (111)$$

Wigner functions

Wigner functions for spin-1/2 particles

$$\begin{aligned} W_{\alpha\beta}(x, k) &= -\frac{1}{(2\pi)^4} \int d^4 y e^{-iky} \left\langle \psi_\alpha \left(x + \frac{y}{2} \right) \bar{\psi}_\beta \left(x - \frac{y}{2} \right) \right\rangle \\ &= \frac{1}{(2\pi)^4} \int d^4 y e^{-iky} \left\langle \bar{\psi}_B \left(x - \frac{y}{2} \right) \psi_A \left(x + \frac{y}{2} \right) \right\rangle \\ &= \int \frac{d^3 p}{(2\pi)^3 2E_p} \left[\delta^4(k - p) u_r(p) f_{rs}(x, p) \bar{u}_s(p) \right. \\ &\quad \left. - \delta^4(k + p) v_r(p) \bar{f}_{rs}^T(x, p) v_s(p) \right] \end{aligned} \quad (112)$$

where

$$\begin{aligned} f_{rs}(x, p) &= \frac{1}{2(2\pi)^3} \int d^4 u \delta(u \cdot p) e^{-iu \cdot x} \left\langle a_{p-u/2,s}^\dagger a_{p+u/2,r} \right\rangle \\ \bar{f}_{sr}(x, p) &= \frac{1}{2(2\pi)^3} \int d^4 u \delta(u \cdot p) e^{-iu \cdot x} \left\langle b_{p-u/2,r}^\dagger a_{p+u/2,s} \right\rangle \end{aligned} \quad (113)$$

So the expectation value of any 4×4 matrix O can be obtained by

$$\begin{aligned}\langle \bar{\Psi} O \Psi \rangle &= \int d^4 k \text{Tr} \langle O W(x, k) \rangle \\ &= \int \frac{d^3 p}{(2\pi)^3 2E_p} \left\{ \text{Tr} [O u_r(p) f_{rs}(x, p) \bar{u}_s(p)] \right. \\ &\quad \left. - \text{Tr} [O v_r(p) \bar{f}_{rs}^T(x, p) \bar{v}_s(p)] \right\} \\ &= \int \frac{d^3 p}{(2\pi)^3 2E_p} \left[\bar{u}_s(p) O u_r(p) f_{rs}(x, p) \right. \\ &\quad \left. - \bar{v}_s(p) O v_r(p) \bar{f}_{rs}^T(x, p) \right] \end{aligned} \quad (114)$$

The 4×4 matrix O corresponding to the current vector, the energy-momentum and spin tensor are

$$\begin{aligned} T^{\mu\nu} &= \frac{i}{2} \gamma^\mu \overleftrightarrow{\partial}^\nu \\ j^\mu &= \gamma^\mu \\ S^{\lambda, \mu\nu} &= \frac{1}{2} \left\{ \gamma^\lambda, \Sigma^{\mu\nu} \right\} \end{aligned} \quad (115)$$

Note that the energy-momentum and spin tensor are all canonical.

Spin distribution functions

According to (109), we have

$$f_{rs} = e^{-\beta \cdot p + \xi} \frac{1}{2} \left[D^{(S)} \left(L^{-1}(p) \hat{R}_\omega L(p) \right)_{rs} + D^{(S)} \left(L^\dagger(p) \hat{R}_\omega \left(L^\dagger(p) \right)^{-1} \right)_{rs} \right] \quad (116)$$

where $\xi = \mu q / T$. The trace of this distribution corresponding to the particle density is

$$\text{Tr}_s f = \sum_s f_{ss} = e^{-\beta \cdot p - \beta \mu q} \text{Tr}_s \left[D^{(S)} \left(\hat{R}_\omega \right) \right] \quad (117)$$

We apply it to spin-1/2 particles as

$$f = e^{-\beta \cdot p + \xi} \frac{1}{2m} \bar{U}(p) \exp \left(\frac{1}{2} \beta \omega \Sigma_z \right) U(p) \quad (118)$$

Spin distribution functions

Here $U(p)$ is a 4×2 matrix

$$U(p) = \sqrt{m} \begin{pmatrix} D^{(1/2)}(L(p)) \\ D^{(1/2)\dagger}(L^{-1}(p)) \end{pmatrix} \quad (119)$$

it has a additional factor $\sqrt{m}$ compared with Eq. (96), and we assume

$$D^{(1/2)}(i\beta\omega) = \exp\left(\frac{1}{2}\beta\omega\sigma_3\right) \quad (120)$$

and

$$\Sigma_z = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \quad (121)$$

Spin distribution functions

We can define

$$\bar{\omega}_{\mu\nu} = \beta\omega (\delta_{\mu}^1\delta_{\nu}^2 - \delta_{\mu}^2\delta_{\nu}^1) \quad (122)$$

So Eq. (118) can be rewritten as

$$\begin{aligned} f &= e^{-\beta \cdot p + \xi} \frac{1}{2m} \bar{U}(p) \exp\left(\frac{1}{2} \bar{\omega}_{\mu\nu} \Sigma^{\mu\nu}\right) U(p) \\ \bar{f} &= -e^{-\beta \cdot p + \xi} \frac{1}{2m} \left[\bar{V}(p) \exp\left(-\frac{1}{2} \bar{\omega}_{\mu\nu} \Sigma^{\mu\nu}\right) V(p) \right]^T \end{aligned} \quad (123)$$

Spin distribution functions

We can make an ansatz that distribution can be expressed as

$$\begin{aligned} f &= \frac{1}{2m} \bar{U}(p) X(p) U(p) \\ \bar{f} &= -\frac{1}{2m} [\bar{V}(p) \bar{X}(p) V(p)]^T \end{aligned} \quad (124)$$

and $X(p)$ and $\bar{X}(p)$ in global equilibrium state are

$$\begin{aligned} X(p) &= \left[\exp \left(\beta \cdot p - \xi - \frac{1}{2} \bar{\omega}_{\mu\nu} \Sigma^{\mu\nu} \right) + I \right]^{-1} \\ \bar{X}(p) &= \left[\exp \left(\beta \cdot p + \xi + \frac{1}{2} \bar{\omega}_{\mu\nu} \Sigma^{\mu\nu} \right) + I \right]^{-1} \end{aligned} \quad (125)$$

Spin tensors

We can construct spin tensors

$$\begin{aligned}\langle \bar{\Psi} S^{\lambda, \mu\nu} \Psi \rangle &= \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3 2E_p} \text{tr}_2 \left[\bar{U}(p) \left\{ \gamma^\lambda, \Sigma^{\mu\nu} \right\} U(p) f(x, p) \right] \\ &\quad - \text{tr}_2 \left[\bar{V}(p) \left\{ \gamma^\lambda, \Sigma^{\mu\nu} \right\} V(p) \bar{f}^T(x, p) \right] \\ &= \int \frac{d^3 p}{(2\pi)^3 2E_p} \left(p^\nu \Theta^{\lambda\mu} + p^\mu \Theta^{\lambda\nu} + p^\lambda \Theta^{\mu\nu} \right. \\ &\quad \left. + p^\nu \bar{\Theta}^{\lambda\mu} + p^\mu \bar{\Theta}^{\lambda\nu} + p^\lambda \bar{\Theta}^{\mu\nu} \right)\end{aligned}$$

where the tensor $\Theta^{\mu\nu}$ and $\bar{\Theta}^{\mu\nu}$ are defined as

$$\begin{aligned}\Theta^{\mu\nu} &= \text{tr} [\Sigma^{\mu\nu} X(p)] \\ \bar{\Theta}^{\mu\nu} &= -\text{tr} [\Sigma^{\mu\nu} \bar{X}(p)]\end{aligned}\tag{126}$$

The tensor $\Theta^{\mu\nu}$ can be written as in form of derivative of $\bar{\omega}^{\mu\nu}$

$$\begin{aligned}
 \Theta^{\mu\nu} &= \text{Tr} \left\{ \Sigma^{\mu\nu} \left[\exp \left(\beta \cdot p - \xi - \frac{1}{2} \bar{\omega}_{\mu\nu} \Sigma^{\mu\nu} \right) + I \right]^{-1} \right\} \\
 &= \sum_{n=1}^{\infty} \text{Tr} \left[\Sigma^{\mu\nu} (-1)^{n+1} \exp \left(-n\beta \cdot p + n\xi + \frac{1}{2} n\bar{\omega}_{\mu\nu} \Sigma^{\mu\nu} \right) \right] \\
 &= 2 \frac{\partial}{\partial \bar{\omega}_{\mu\nu}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \exp(-n\beta \cdot p + n\xi) \text{Tr} \left[\exp \left(\frac{1}{2} n\bar{\omega}_{\mu\nu} \Sigma^{\mu\nu} \right) \right] \\
 &= 2 \frac{\partial}{\partial \bar{\omega}_{\mu\nu}} \text{Tr} \left[\ln \left(I + \exp \left(-\beta \cdot p + \xi + \frac{1}{2} \bar{\omega}_{\mu\nu} \Sigma^{\mu\nu} \right) \right) \right] \\
 &\approx n_F (1 - n_F) \bar{\omega}^{\mu\nu}
 \end{aligned} \tag{127}$$

The Pauli-Lubanski vector is defined as

$$S_\mu = -\frac{1}{2m}\epsilon_{\mu\rho\sigma\tau}J^{\rho\sigma}p^\tau \quad (128)$$

where $J^{\rho\sigma}$ are generators of Poincare algebra corresponding to Lorentz boost and rotation and are given by

$$J^{\rho\sigma} = (2\pi)^3 \frac{d\mathcal{J}^{0,\rho\sigma}(x, p)}{d^3p} \quad (129)$$

The total angular momentum operator is defined as

$$\mathcal{J}^{\lambda,\rho\sigma}(x, p) = x^\rho T^{\lambda\sigma} - x^\sigma T^{\lambda\rho} + \mathcal{S}^{\lambda,\rho\sigma} \quad (130)$$

The stress tensor and density are defined as

$$\begin{aligned} T^{\mu\nu} &= \int \frac{d^3p}{(2\pi)^3 2E_p} p^\mu p^\nu (\text{tr}_2 f + \text{tr}_2 \bar{f}) \\ \mathcal{S}^{\lambda,\mu\nu} &= \int \frac{d^3p}{(2\pi)^3 2E_p} \left(p^\nu \Theta^{\lambda\mu} + p^\mu \Theta^{\lambda\nu} + p^\lambda \Theta^{\mu\nu} \right. \\ &\quad \left. + p^\nu \bar{\Theta}^{\lambda\mu} + p^\mu \bar{\Theta}^{\lambda\nu} + p^\lambda \bar{\Theta}^{\mu\nu} \right) \end{aligned} \quad (131)$$

For the particle, we have

$$\begin{aligned}
 \mathcal{S}^{0,\rho\sigma} &= \frac{1}{2} [(x^\rho p^\sigma - x^\sigma p^\rho) \text{tr}_2 f + (x^\rho p^\sigma - x^\sigma p^\rho) \text{tr}_2 \bar{f}] \\
 &\quad + \frac{1}{2E_p} (p^\sigma \Theta^{0\rho} + p^\rho \Theta^{0\sigma} + p^0 \Theta^{\rho\sigma} \\
 &\quad + p^\sigma \bar{\Theta}^{0\rho} + p^\rho \bar{\Theta}^{0\sigma} + p^0 \bar{\Theta}^{\rho\sigma}) \\
 &\quad \downarrow \epsilon_{\mu\rho\sigma\tau} \mathcal{S}^{0,\rho\sigma} \\
 &= \frac{1}{2} (\Theta^{\rho\sigma} + \bar{\Theta}^{\rho\sigma})
 \end{aligned} \tag{132}$$

the terms proportional to p^ρ or p^σ vanish for the

$\epsilon_{\mu\rho\sigma\tau} p^\tau p^\rho = \epsilon_{\mu\rho\sigma\tau} p^\tau p^\sigma = 0$. The average polarization defined in Eq.(128) can be simplified to

$$\langle S_\mu(p) \rangle = -\frac{1}{4\text{tr}_2 f} \epsilon_{\mu\rho\sigma\tau} \Theta^{\rho\sigma} \frac{p^\tau}{m} \tag{133}$$

Spin vector on hyper-surface

The momentum spectra can be obtained by integration over a volume V on freezeout hyper-surface

$$\begin{aligned}\langle S_\mu(x, p) \rangle &= -\frac{1}{4} \frac{\int_V d^3x \epsilon_{\mu\rho\sigma\tau} \Theta^{\rho\sigma} \frac{p^\tau}{m}}{\int_V d^3x \text{tr}_2 f} \\&= -\frac{1}{4} \frac{\int_{\partial V} d\Sigma_\lambda \frac{p^\lambda}{E} \epsilon_{\mu\rho\sigma\tau} \Theta^{\rho\sigma} \frac{p^\tau}{m}}{\int_{\partial V} d\Sigma_\lambda \frac{p^\lambda}{E} \text{tr}_2 f} \\&= -\frac{1}{4} \epsilon_{\mu\rho\sigma\tau} \frac{p^\tau}{m} \frac{\int_{\partial V} d\Sigma_\lambda p^\lambda \Theta^{\rho\sigma}}{\int_{\partial V} d\Sigma_\lambda p^\lambda \text{tr}_2 f} \\&\simeq -\frac{1}{8} \epsilon_{\mu\rho\sigma\tau} \frac{p^\tau}{m} \frac{\int_{\partial V} d\Sigma_\lambda p^\lambda n_F (1 - n_F) \bar{\omega}^{\rho\sigma}}{\int_{\partial V} d\Sigma_\lambda p^\lambda n_F} \quad (134)\end{aligned}$$

Spin vector on hyper-surface

Here the hypersurface $d\Sigma_\lambda = d^3x \hat{t}_\mu$, the $\hat{t}_\mu$ is the normal vector of the hypersurface. The polarization for anti-particles can be obtained by replacing f and $\Theta^{\mu\nu}$ by $\bar{f}$ and $\bar{\Theta}^{\mu\nu}$ respectively.