# A 2D-CFT FACTORY: CRITICAL LATTICE MODELS FROM COMPETING ANYON CONDENSATION IN SYMTO

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## Motivation and Background: How to Construct a Critical Lattice Model?

### STRANGE CORRELATOR

#### TOPOLOGICAL HOLOGRAPHIC PRINCIPLE

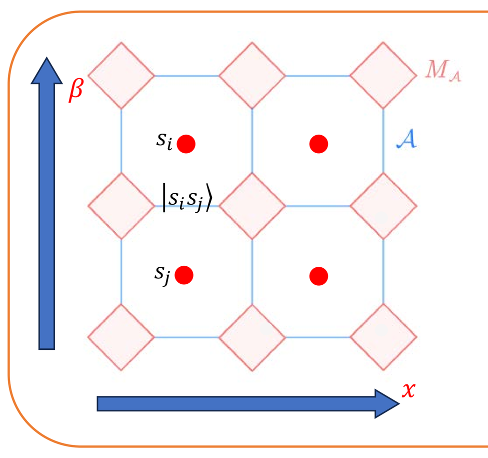
*ArXiv: 1912.13492*

Path integral of $d$-$D$ QFT with symmetry described by fusion category $\mathcal{F}$
~ path integral of $(d+1)$-D TQFT (SymTO) with input fusion category $\mathcal{F}$

Strange correlator is an explicit realization

$Z(\beta) = \langle \Omega(\beta) | \Psi \rangle$ —*RG Fixed Points*→

- **Gapped: SPTs**
- **Gapless: CFTs**

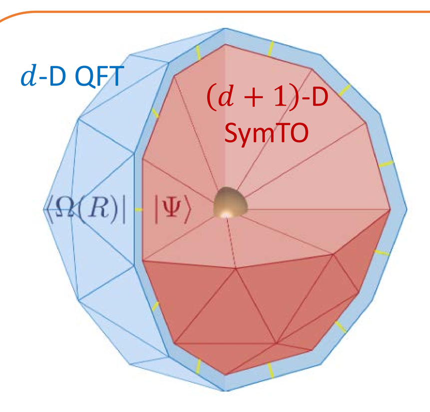



Red dots: Ising spins $1, -1$

Octagon edges: $|1\rangle, |-1\rangle$

Square edges: $|1\rangle + |-1\rangle$

$$\langle \Omega(\beta)| = \sum_{\{\sigma_i = 0,1\}} \bigotimes_{\text{Edges}:\langle ij \rangle} (e^{\beta}\langle 0| + e^{-\beta}\langle 1|)$$




$$|\text{String-Net GS}\rangle = \sum_{\{\sigma_i = 0,1\}} \bigotimes_{\text{Edges}:\langle ij \rangle} |s_i s_j\rangle$$

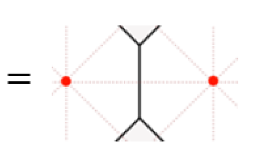
$Z(\beta) = \langle \Omega(\beta) | \text{String-Net GS} \rangle$

How to find critical $\langle \Omega |$?

### COMPETING CONDENSATES IN TORIC CODE

Transfer Matrix:

$$|\Omega(r)\rangle = \begin{pmatrix} e^{\beta} \\ e^{-\beta} \end{pmatrix} =$$




- $\beta = \infty \Longrightarrow$ *Order* ($Z_2$ symmetry breaking)
- $\beta = 0 \Longrightarrow$ *Disorder* (dual $Z_2$ symmetry breaking)
- $\beta = \frac{1}{2}\ln(\sqrt{2}+1) \Longrightarrow$ Critical!

Toric Code Model

- Chargeon condensation
  $W_e = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
  $|\Omega\rangle_e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- Fluxon condensation
  $W_m = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
  $|\Omega\rangle_m = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
- Competing condensates
  $|\Omega\rangle_e + |\Omega\rangle_m \sim \begin{pmatrix} \sqrt{2}+1 \\ 1 \end{pmatrix}$

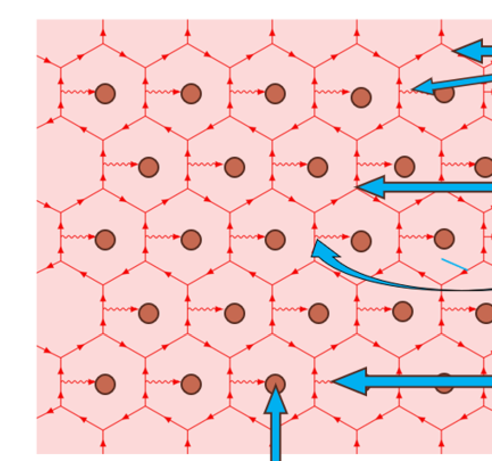
$$H(a,b) = H_{\text{Toric code}} - \Lambda(aW_e + bW_m), \qquad \Lambda \to +\infty$$

- $a = 0$ or $b = 0$: chargeon/fluxon condensation $\Longrightarrow$ Gapped (SPT) States
- $a > b$: More chargeons than fluxons $\Longrightarrow$ Gapped States after RG flow
- $a < b$: More fluxons than chargeons $\Longrightarrow$ Gapped States after RG flow
- $a = b$: ***Equally matched*** $\Longrightarrow$ **Gapless Ising CFT preserving $\mathbb{Z}_2$ symmetry!**

### CONDENSATION IN MODEL

#### HU-GEER-WU STRING-NET MODEL

*ArXiv: 2506.05319*



Edges/Tails: **Gauge** Dofs
E.g., $\{1, \psi, \sigma\}$ in Ising Fusion category

Vertices: Always satisfying Fusion Rules!
*Topological Gauge Fields are always Gauge Flat!*

Tail Endpoint: Guage Curvature:
E.g., $1, \psi, \sigma$ in Ising Fusion category

***Anyons*** E.g.: $1\bar{1}, 1\bar{\sigma}, 1\bar{\psi}, \psi\bar{1}, \psi\bar{\sigma}, \psi\bar{\psi}, \sigma\bar{1}, \sigma\bar{\sigma}, \sigma\bar{\psi},$

E.g.: $(\sigma\bar{\sigma}, 1), (\sigma\bar{\sigma}, \psi)$

E.g.: 2D irrep of $D(S_3)$

- *Anyon Type: Simple objects in Drinfeld Center of Input Fusion Category*
- *Flux (Monopole): Providing curvature via magnetic Gauss Law*
- *Internal Charge Space: Gauge UFC Representation Space*

#### FULL ANYON CONDENSATION

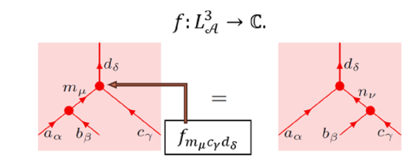
*ArXiv: 2409.05852*

Each edge of the lattice is fixed to take a ***Frobenius algebra*** object $\mathcal{A}$ of the input fusion category

*Frobenius algebra:* $\mathcal{A} = (L_{\mathcal{A}}, f)$:

$$L_{\mathcal{A}} = \{a_\alpha | a \in \text{Irr}(\mathcal{F}), 1 \le \alpha \le n_a^{\mathcal{A}}\},$$

$$f: L_{\mathcal{A}}^3 \to \mathbb{C}.$$




$Z(\mathcal{A})$: a Lagrangian algebra of $Z(\mathcal{F})$
condensed anyon types

Example: Ising $\{1, \sigma, \psi\}$

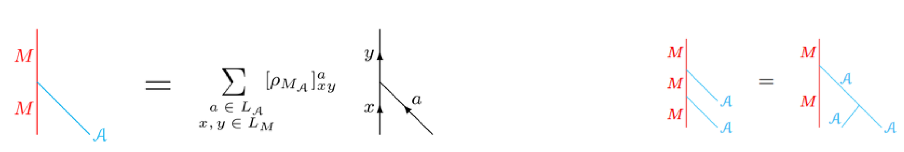
| Frobenius algebra | $\mathcal{A}_0$ | $\mathcal{A}_1$ |
|---|---|---|
| $L_{\mathcal{A}}$ | $\{1\}$ | $\{1, \psi\}$ |
| Condensed Anyons | $1\bar{1}, \psi\bar{\psi}, \sigma\bar{\sigma}$ | $1\bar{1}, \psi\bar{\psi}, \sigma\bar{\sigma}$ |
| Condensed Sectors | $1\bar{1}, \psi\bar{\psi}, (\sigma\bar{\sigma}, 1)$ | $1\bar{1}, \psi\bar{\psi}, (\sigma\bar{\sigma}, \psi)$ |

$\mathcal{A}_0$ and $\mathcal{A}_1$ condense the same anyon types!
*But they are **physically different**!*
*Can compete and lead to phase transitions!*

## Recipe: Via Competing Anyon Condensations in the String-Net Model

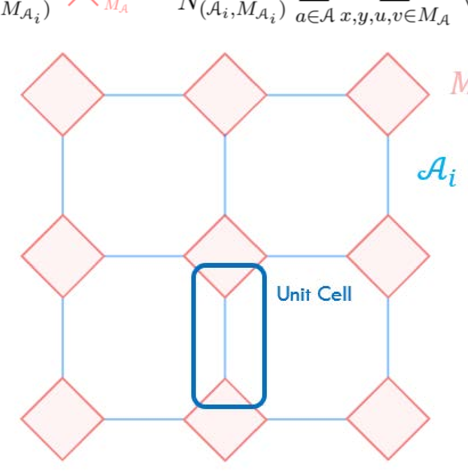
### SINGLE CONDENSATE ON UNIT CELL

- Given an HGW string-net model with input *fusion category* $\mathcal{F}$
- Select a condensate described by a *Frobenius algebra* $\mathcal{A}_i$ of $\mathcal{F}$
- Place *Frobenius algebra* $\mathcal{A}_i$ on each octagon edge
- Place a *module* $M_{\mathcal{A}_i}$ of $\mathcal{A}_i$ on each square edge to absorb tails and *separate* global condensation into unentangled local condensates



- Resultant normalized child state on a unit cell:

$$\langle \hat{\mathcal{A}}_i |_{M_{\mathcal{A}_i}} := \frac{1}{N_{(\mathcal{A}_i, M_{\mathcal{A}_i})}} \langle \cdots | = \frac{1}{N_{(\mathcal{A}_i, M_{\mathcal{A}_i})}} \sum_{a \in \mathcal{A}} \sum_{x,y,u,v \in M_{\mathcal{A}}} \langle \cdots | \, [\rho_{M_{\mathcal{A}}}]^a_{xy} ([\rho_{M_{\mathcal{A}}}]^a_{uv})^*$$




### COMPETING CONDENSATE

- Select ***two*** anyon condensate described by Frobenius algebras $\mathcal{A}_i$ and $\mathcal{A}_j$
- Select ***common module*** $M_{\mathcal{A}_{i,j}}$ of $\mathcal{A}_i$ and $\mathcal{A}_j$
- Construct the normalized child states $\langle \hat{\mathcal{A}}_i |, \langle \hat{\mathcal{A}}_j |$
- **Ansatz: Critical states are located where all child states are equally matched!**

$$\langle \Omega |_{critical} = \langle \bigtimes_{(\mathcal{A}_i, M_{\mathcal{A}_i}), (\mathcal{A}_j, M_{\mathcal{A}_j})} |_{\text{critical}} = \langle \hat{\mathcal{A}}_i |_{M_{\mathcal{A}_i}} + \langle \hat{\mathcal{A}}_j |_{M_{\mathcal{A}_j}}$$

- Using ***Strange Correlator*** to obtain ***Critical Partition function***:

$$Z = \langle \Omega | \text{StringNet GS} \rangle$$

- Competing of three anyon condensate $\mathcal{A}_i, \mathcal{A}_j$, and $\mathcal{A}_k$?
  Phase boundary between $\mathcal{A}_i$ and $\mathcal{A}_j$: Schmidt orthogonalization!

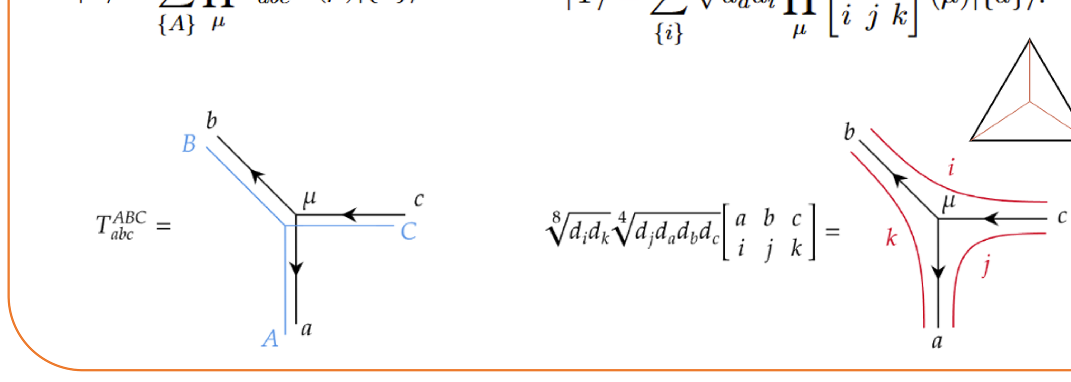
$$|\Omega_{ij}(\lambda)\rangle = |\Omega_{ij}\rangle + \lambda\left(|\mathcal{A}_k\rangle - \frac{\langle \Omega_{ij} | \mathcal{A}_k \rangle}{\langle \Omega_{ij} | \Omega_{ij} \rangle} |B_{ij}\rangle\right) \qquad |B_{ij}\rangle = |\mathcal{A}_i\rangle - |\mathcal{A}_j\rangle$$

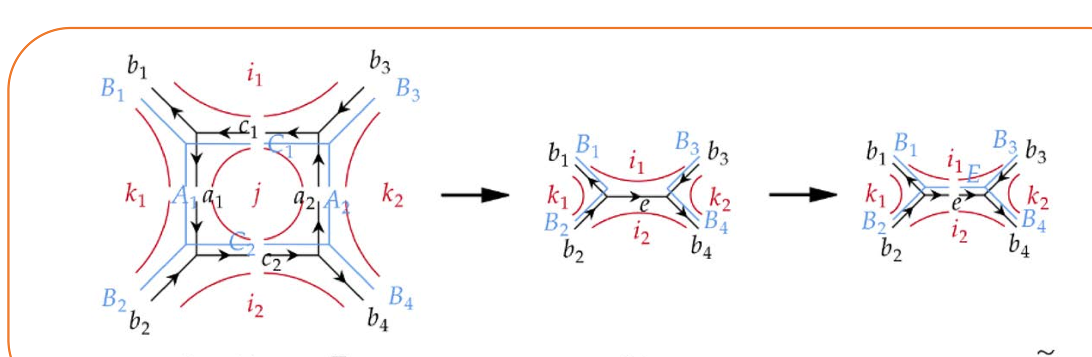
Likewise phase boundaries $|\Omega_{ik}(\lambda)\rangle, |\Omega_{jk}(\lambda)\rangle$, and the ***tricritical point***!

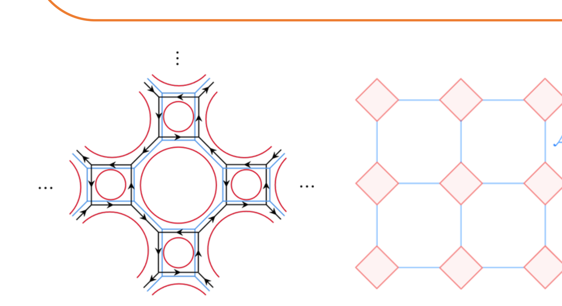
### RENORMALIZATION GROUP FLOW

Tensor products of unit cells are no longer RG Fixed Points $\Longrightarrow$ Need RG.

$$|\Omega\rangle = \sum_{\{A\}} \prod_{\mu} T_{abc}^{ABC}(\mu) |\{a\}\rangle \qquad |\Psi\rangle = \sum_{\{i\}} \sqrt{d_a} d_i \prod_{\mu} \begin{bmatrix} a & b & c \\ i & j & k \end{bmatrix} (\mu) |\{a\}\rangle.$$



$$\lambda_k = e^{2i\pi\tau\left(\Delta_k - \frac{c}{12}\right)}$$



## Examples

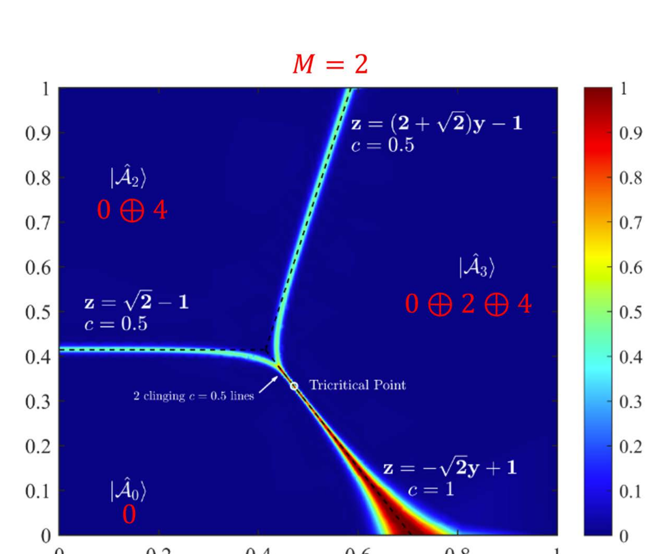
### $\mathbb{Z}_n$ FUSION CATEGORY → POTTS CFTS

- $\mathbb{Z}_n$ Fusion Category: $0, 1, 2, \ldots, n-1$
- Frobenius algebras: $\mathcal{A}_i = 0 \qquad \mathcal{A}_j = \oplus_{i=0}^{n-1} i$
- Common module: $M = \oplus_{i=0}^{n-1} i$

| Cat | $Z_2$ | $Z_3$ | $Z_4$ | | | $Z_5$ |
|---|---|---|---|---|---|---|
| $\mathcal{A}_i$ | $0$ | $0$ | $0$ | | $0 \oplus 2$ | $0$ |
| $\mathcal{A}_j$ | $0 \oplus 1$ | $0 \oplus 1 \oplus 2$ | $0 \oplus 2$ | $0 \oplus 1 \oplus 2 \oplus 3$ | | $0 \oplus 1 \oplus 2 \oplus 3 \oplus 4$ |
| $M$ | $0 \oplus 1$ | $0 \oplus 1 \oplus 2$ | $0 \oplus 2$ | $0 \oplus 1 \oplus 2 \oplus 3$ | | $0 \oplus 1 \oplus 2 \oplus 3 \oplus 4$ |
| $D_{\text{unit}}$ | 2 | 3 | 2 | 4 | 4 | 5 |
| Type | Second order | | | | | First order |
| CFT | Ising $c = 0.5$ | 3-Potts $c = 0.8$ | Ising $c = 0.5$ | 4-Potts $c = 1$ | Ising $c = 0.5$ | Possibly a complex CFT |

### $Rep(S_3)$ CATEGORY

- $Rep(S_3)$ Fusion Category: $0, 2, 4$
  $2 \otimes 2 = 0 \oplus 2 \oplus 4, \quad 2 \otimes 4 = 4$
- Frobenius algebras:
  $\mathcal{A}_0 = 0, \qquad \mathcal{A}_1 = 0 \oplus 2,$
  $\mathcal{A}_2 = 0 \oplus 2 \oplus 4$
- Common module: $M = 2$




### HAAGERUP CATEGORY → NOVEL CFTS

- Haagerup Fusion Category: $1, \alpha, \alpha^2, \rho, \alpha\rho, \alpha^2\rho$
  $\alpha^3 = 1, \qquad \rho\alpha = \alpha^2\rho, \qquad \rho^2 = 1 \oplus \rho \oplus \alpha\rho \oplus \rho\alpha$
- Frobenius algebras:
  $\mathcal{A}_0 = 1, \qquad \mathcal{A}_1 = 1 \oplus \alpha\rho \oplus \alpha^2\rho \oplus \rho, \qquad \mathcal{A}_2 = 1 \oplus \alpha \oplus \alpha^2,$
  $\mathcal{A}_4 = 1 \oplus \rho \oplus \alpha\rho, \qquad \mathcal{A}_5 = 1 \oplus \rho \oplus \alpha^2\rho, \qquad \mathcal{A}_6 = 1 \oplus \alpha\rho \oplus \alpha^2\rho$
- Common module: $M = \rho \oplus \alpha\rho \oplus \alpha^2\rho$

| $\mathcal{A}_i, \mathcal{A}_j(, \mathcal{A}_k)$ | $M$ | $D_c$ | Type* | Sym** | $C$ |
|---|---|---|---|---|---|
| $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ | $\rho \oplus \alpha\rho \oplus \alpha^2\rho$ | 5 | Cont. | $\mathcal{Z}(H_3)$ | 0.793 |
| $\mathcal{A}_0, \mathcal{A}_2, \mathcal{A}_{(4,5,6)}$ | $\rho \oplus \alpha\rho \oplus \alpha^2\rho$ | 4 | Cont. | $\mathcal{Z}(H_3)$ | 0.8047 |
| $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_{(4,5,6)}$ | $\rho \oplus \alpha\rho \oplus \alpha^2\rho$ | 5 | Cont. | $\mathcal{Z}(H_3)$ | 1.3154 |
| $\mathcal{A}_0, \mathcal{A}_{(4,5,6)}, \mathcal{A}_{(5,6,4)}$ | $\rho \oplus \alpha\rho \oplus \alpha^2\rho$ | 4 | Cont. | $\mathcal{Z}(H_3)$ | 1.3141 |
| $\mathcal{A}_1, \mathcal{A}_{(4,5,6)}, \mathcal{A}_{(5,6,4)}$ | $\rho \oplus \alpha\rho \oplus \alpha^2\rho$ | 7 | First | — | — |
| $\mathcal{A}_2, \mathcal{A}_{(4,5,6)}, \mathcal{A}_{(5,6,4)}$ | $\rho \oplus \alpha\rho \oplus \alpha^2\rho$ | 6 | First | — | — |
| $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_{(4,5,6)}$ | $\rho \oplus \alpha\rho \oplus \alpha^2\rho$ | 7 | First | — | — |
| $\mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6$ | $\rho \oplus \alpha\rho \oplus \alpha^2\rho$ | 6 | Cont. | $\mathcal{Z}(H_3)$ | 2.106 |