Multi-entropy from Linking in Chern-Simons Theory

Ma-Ke Yuan, Mingyi Li, and Yang Zhou

1. Multi-entropy

Entanglement entropy (EE) & Rényi entropy

EE: bipartite entanglement measure

$$S_1^{(2)}(A;B) = S_{\mathrm{EE}}(A;B) = -\mathrm{Tr}(\rho_A \log \rho_A)$$

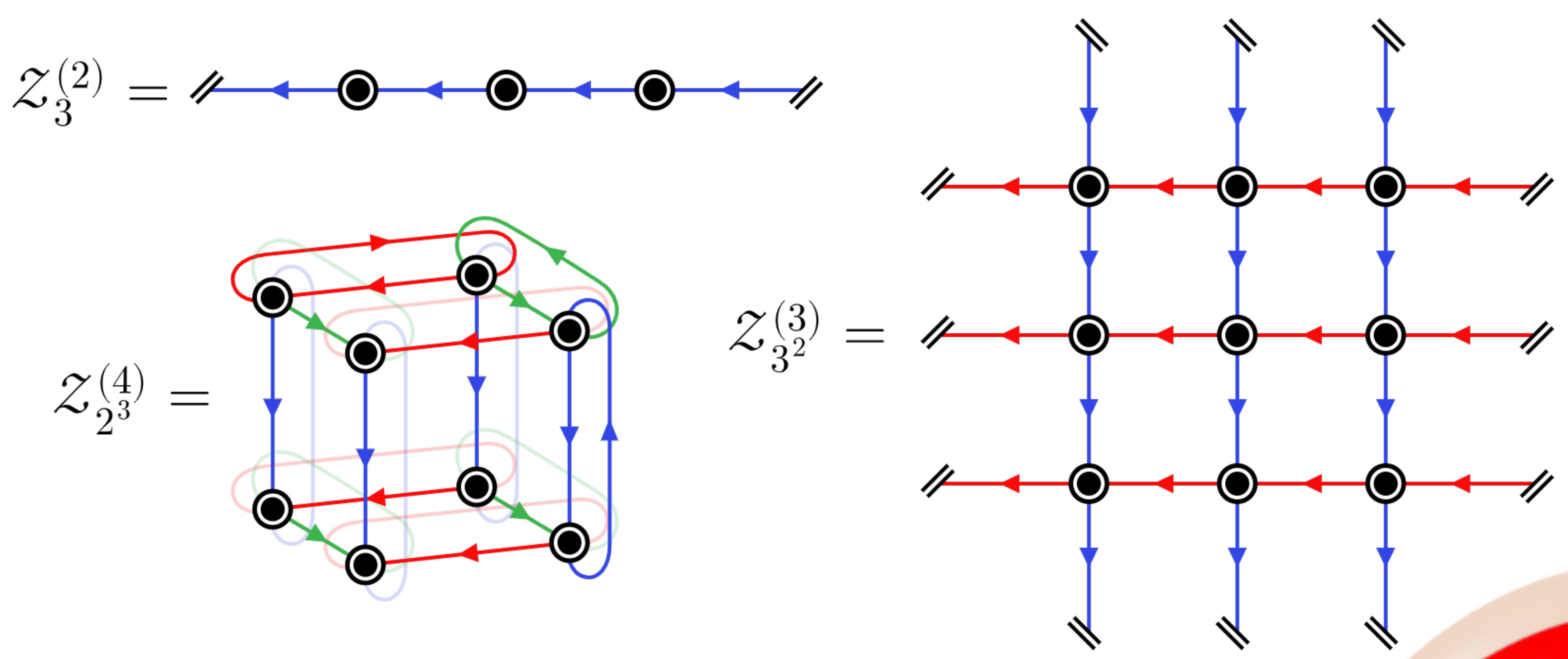
Rényi entropy: one-parameter generalization of EE

$$S_n^{(2)}(A;B) = \frac{1}{1-n}\log \mathrm{Tr}\rho_A^n = \frac{1}{1-n}\log \mathcal{Z}_n$$

Rényi multi-entropy & Multi-entropy

Multipartite generalization of Rényi entropy & EE

$$S_n^{(\mathrm{q})}(A;B;C;\cdots) = \frac{1}{1-n}\frac{1}{n^{\mathrm{q}-2}}\log \mathcal{Z}_{n^{\mathrm{q}-1}}^{(\mathrm{q})} \overset{n\to 1}{\Longrightarrow} S_1^{(\mathrm{q})}(A;B;C;\cdots)$$



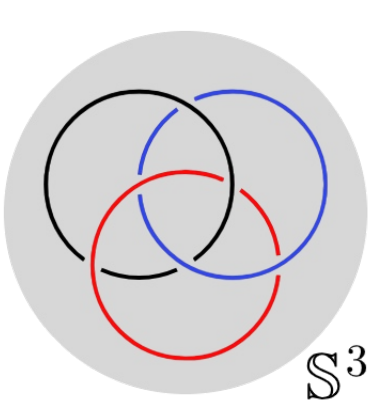
2. Link states

Chern-Simons theory

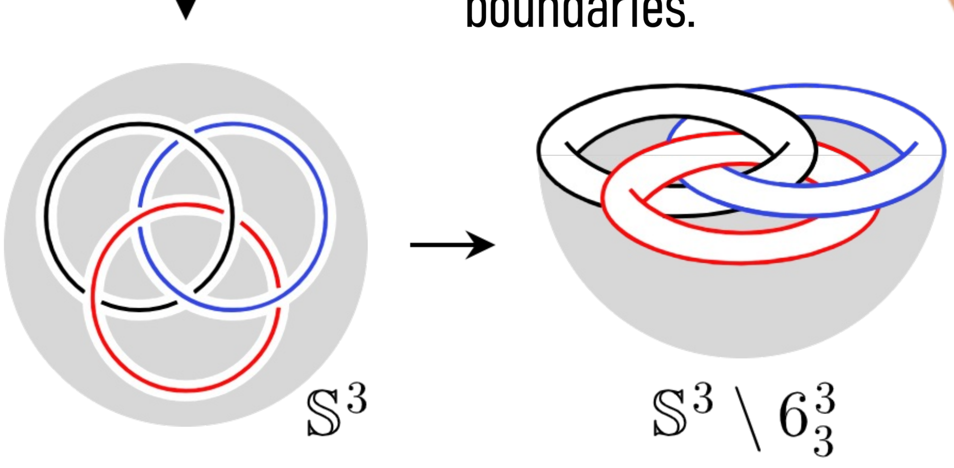
CS action with gauge field A:

$$\frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{Tr}\left(A\wedge \mathrm{d}A + \frac{2}{3}A\wedge A\wedge A\right)$$

Link states from path integral



- Start with a 3-sphere
- Remove a tubular neighborhood of the link (here 6_3^3)
- We get it! A link state living on the torus boundaries.



Wave function of 3-component links in $\mathrm{U}(1)_k$ CS

$$|\mathcal{L}^3\rangle = \frac{1}{k^{3/2}}\sum_{\alpha,\beta,\chi}^{k-1}\exp\left(\frac{2\pi i}{k}(\alpha\beta L_{AB} + \beta\chi L_{BC} + \chi\alpha L_{CA})\right)|\alpha,\beta,\chi\rangle$$

3. Main results

Closed formula for Rényi multi-entropy

$$S_n^{(3)}(A;B;C) = \frac{1}{n}\log\frac{k^3}{\gcd(k,L_{CA},L_{AB})\gcd(k,L_{AB},L_{BC})\gcd(k,L_{BC},L_{CA})} + \left(1-\frac{2}{n}\right)\log\frac{k}{\gcd(k,L_{CA},L_{AB},L_{BC})}$$

Genuine multi-entropy

$$\mathrm{GM}_n^{(3)}(A;B;C) \equiv S_n^{(3)} - \frac{1}{2}\left(S_n^{(2)}(A) + S_n^{(2)}(B) + S_n^{(2)}(C)\right) = \left(\frac{1}{n}-\frac{1}{2}\right)\log\frac{k\cdot \gcd(k,L_{CA},L_{AB},L_{BC})^2}{\gcd(k,L_{CA},L_{AB})\gcd(k,L_{AB},L_{BC})\gcd(k,L_{BC},L_{CA})}$$

Logarithmic negativity

$$E_{\mathcal{N}}(B;C) = \lim_{p\to 1/2}\log\left(\rho_{BC}^{\Gamma}\right)^{2p} = \log\frac{\gcd(k,L_{CA},L_{AB})}{\gcd(k,L_{CA},L_{AB},L_{BC})}$$

4. Physical meanings

Link states in Abelian CS theory are *stabilizer states.*

Prime-power qudit stabilizer states

$\forall$ *N*-qudit stabilizer state with prime-power dimension $k = p^{\epsilon}$,
$\exists$ local unitary $U = U_A \otimes U_B \otimes U_C$ s.t.

$$U|\mathrm{S}_{p^\epsilon}\rangle_{ABC} = |\mathrm{GHZ}\rangle_{ABC}^{\otimes g} \otimes |\mathrm{EPR}\rangle_{AB}^{\otimes c} \otimes |\mathrm{EPR}\rangle_{BC}^{\otimes a} \otimes |\mathrm{EPR}\rangle_{CA}^{\otimes b}$$

with GHZ and EPR given by

$$|\mathrm{GHZ}\rangle = \textstyle\sum_{i=0}^{p-1}|iii\rangle\,,\quad |\mathrm{EPR}\rangle = \sum_{i=0}^{p-1}|ii\rangle$$

Genuine multi-entropy extracts # of GHZ

$$2\mathrm{GM}_1^{(3)}[\mathrm{S}_{ABC}] = g\log p$$

Logarithmic negativity extracts # of EPR

$$E_{\mathcal{N}}(B;C)[\mathrm{S}_{ABC}] = a\log p$$

General qudit stabilizer states

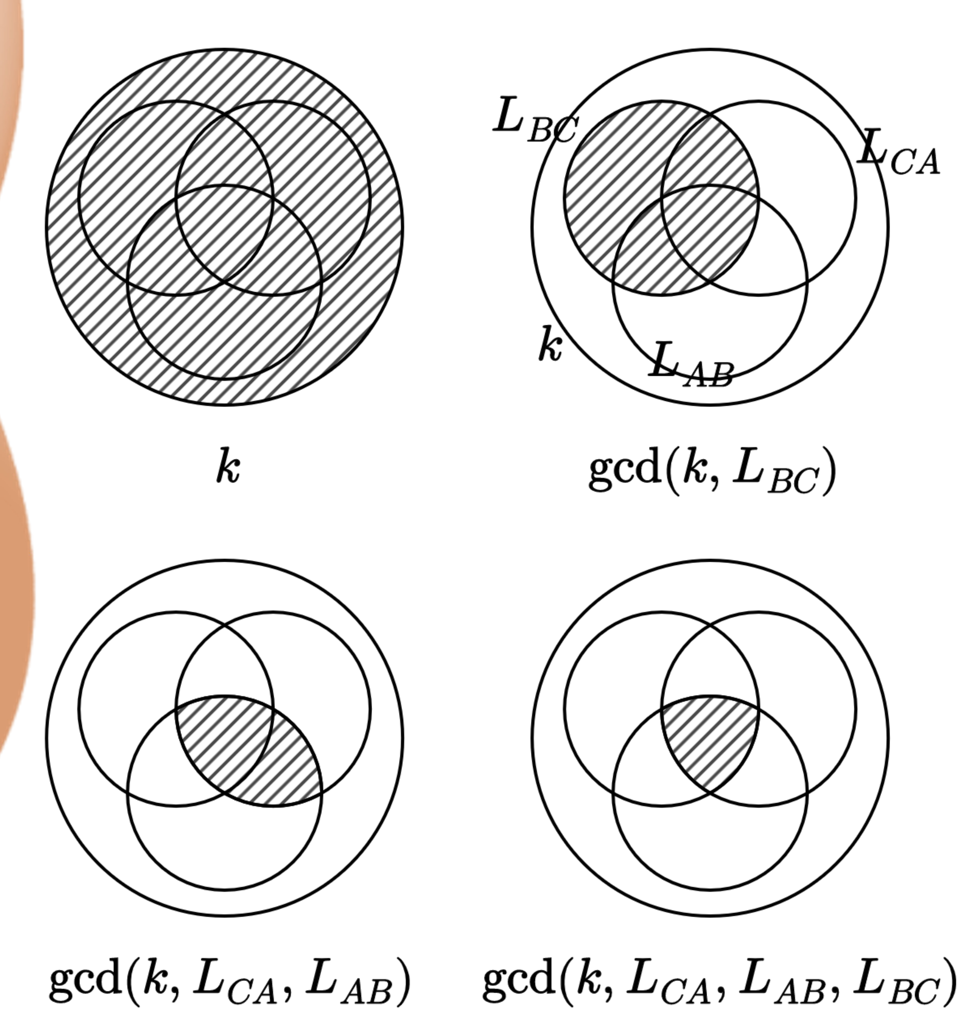
For a general integer $k = p_1^{\epsilon_1}p_2^{\epsilon_2}\cdots p_\ell^{\epsilon_\ell}$, we have

$$U|\mathrm{S}_k\rangle_{ABC} = \otimes_{i=1}^{\ell}|\mathrm{S}_i\rangle_{ABC}$$

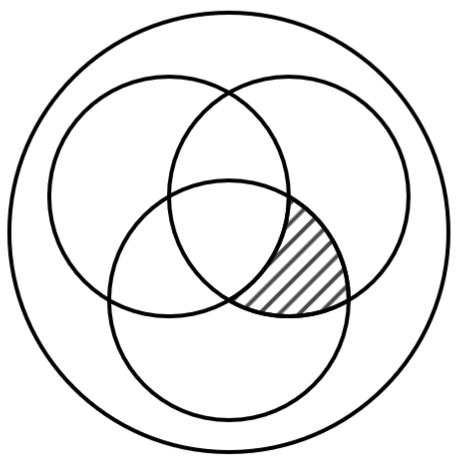
(with $|\mathrm{S}_i\rangle$ a stabilizer state on *N* qubit of dim $p_i^{\epsilon_i}$)

$$\Longrightarrow \quad 2\mathrm{GM}_1^{(3)}[\mathrm{S}_{ABC}] = \textstyle\sum_{i=0}^{\ell} g_i\log p_i$$
$$E_{\mathcal{N}}(B;C)[\mathrm{S}_{ABC}] = \textstyle\sum_{i=0}^{\ell} a_i\log p_i$$

5. Venn diagram



$E_{\mathcal{N}}(B;C) =$



$2\mathrm{GM}_1^{(3)} =$