Fracton Phase Transition via Non-Abelian P-loop Condensation

—— A Cage-Net Model Realization

ABSTRACT Fracton orders in 3+1d hold profound potential for robust quantum memory, and can be built from 2+1d topological orders via p-loop condensation. Nevertheless, condensing p-loops made of non-Abelian quasiparticles remains a theoretical bottleneck. We resolve this by developing a Hamiltonian framework for non-Abelian p-loop, fracton, and planon condensation in 3+1d. By incorporating Hu-Geer-Wu string-net formulations into extended cage-net models, we explicitly resolve the internal gauge spaces of quasiparticles. We demonstrate this framework by condensing non-Abelian loops to drive a phase transition from the Ising cage-net model to the X-cube model. Our results provide a universal mechanism for exploring phase transitions and dimensional decoupling between distinct 3+1d fracton orders and 2+1d topological orders.

arXiv:2604.24755, Yifei Wang, Yu Zhao, Yingchen Li, Yidun Wan and Hao Song(ITP-CAS)

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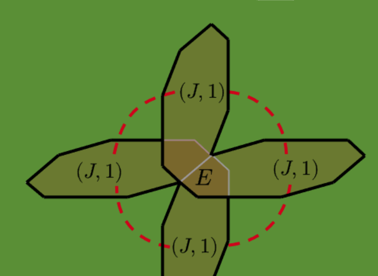
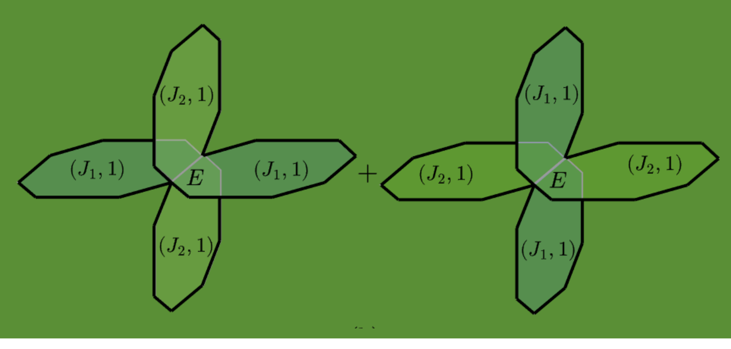
Generalized p-loop condensation term

$$P_E := P_E^{\text{loop}} + P_E^{\text{cross}}$$

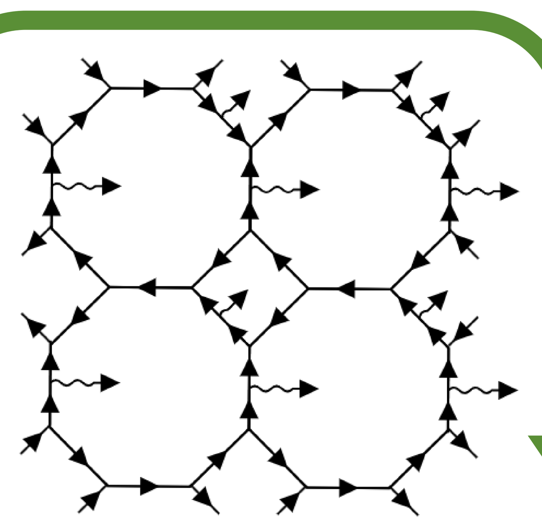
$$P_E^{\text{loop}} = \sum_{\{J\}} \pi_{JJ} W_E^J V_E^J$$



$$P_E^{\text{cross}} = \sum_{J_1,J_2\in\{J\},\, J_1\neq J_2} \pi_{J_1 J_2}\left(W_E^{J_1}V_E^{J_2} + W_E^{J_2}V_E^{J_1}\right)$$



Anyon condensation



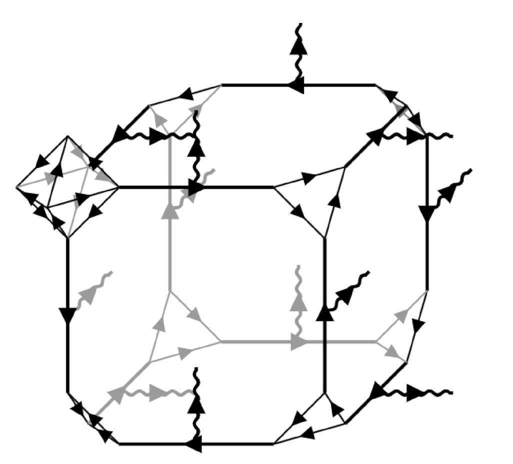
2+1d Topological Order

- Anyons (fractional statistics)
- GSD depend only on topology
- TEE (a constant correction to area law)
- Defined on truncated square lattice

P-loop condensation ↓ Fracton condensation ↑

3+1d Fracton Order

- Fractons (inability to move in isolation)
- Subextensive GSD depend on system size
- Size-dependent correction term
- Defined on truncated cubic lattice



P-loop condensation

Example: Ising TO/FO Hierarchy

Ising HGW model

- d.o.f. $\{1,\psi,\sigma\}$
- Hamiltonian $H_{\text{DI}} = -\sum_p Q_p^{\text{DI}}$
- Anyons $1\bar1, 1\bar\psi, \psi\bar1, \psi\bar\psi, 1\bar\sigma, \sigma\bar1, \psi\bar\sigma, \sigma\bar\psi, \sigma\bar\sigma$

$\psi\bar\psi$ – loop

$(\sigma\bar\sigma, 1)$ – anyon

Ising cage-net model

- d.o.f.(principal edge) $\{1|1, 1|\psi, \psi|1, \psi|\psi, \sigma|\sigma\}$
- Hamiltonian $H_{\text{ICN}} = -\sum_{(E,\mu)}\sum_{K_d} Q_{P_d} - K_P \sum_{(E,\mu)}\sum_{P_o}\left(Q^1_{P_o} + Q^\psi_{P_o}\right) - K_C\sum_C F_C$
- Fracton $\mathfrak{f}_{\psi\bar\psi}$
- Planons $\mathfrak{p}_{1\bar\psi},\ \mathfrak{p}_{\psi\bar1},\ \mathfrak{p}_{\sigma\bar\sigma}$
- Lineons $\mathfrak{l}_{\sigma\bar1,\sigma\bar1},\ \mathfrak{l}_{\sigma\bar1,\sigma\bar\psi},\ \mathfrak{l}_{1\bar\sigma,1\bar\sigma},\ \mathfrak{l}_{1\bar\sigma,\psi\bar\sigma},\ \mathfrak{l}_{1\bar\sigma,\sigma\bar1},\ \mathfrak{l}_{1\bar\sigma,\sigma\bar\psi}$

$\psi\bar\psi$ – anyon

$\mathfrak{p}_{\sigma\bar\sigma,1}$ – planon

$\mathfrak{f}_{\psi\bar\psi}$ – fracton

$(\sigma\bar\sigma, 1)$ – loop

Z_2 toric code model

- d.o.f. $\{1,\psi\}$
- Hamiltonian $H_{\text{TC}} = -\sum_p Q_p^{\text{TC}}$
- Anyons $1, e, m, \epsilon$

e – loop

Z_2 X-cube model

- d.o.f.(principal edge) $\{1|1, \psi|\psi\}$
- Hamiltonian $H_{\text{XC}} = -\sum_{(E,\mu)}\left(\sum_{P_d} Q^{\text{XC}}_{P_d} - \sum_{P_o} Q^1_{P_o}\right) - \sum_C F_C^{\text{XC}}$
- Fractons $\mathfrak{f}_e$
- Lineons $\mathfrak{l}_{mm},\ \mathfrak{l}_{m\epsilon}$

e – anyon

$\mathfrak{f}_e$ – fracton

Vacuum