Positivity bounds in scalar-QED EFT at one-loop level

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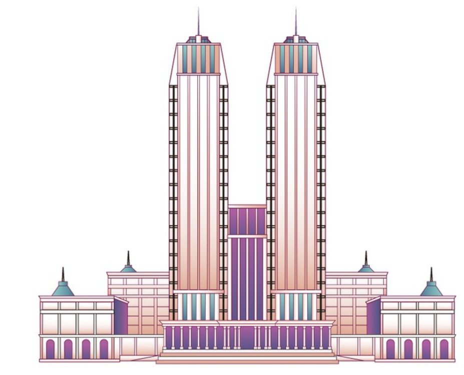

Dispersion Relation

Fundamental principles of QFT—causality, as embodied in the analyticity and crossing properties of amplitude, and unitarity implied by optical theorem constrain certain Wilson coefficients in EFT to be positive[1]. These constraints are referred to as **positivity bounds**. Using the countour in Fig.1 and optical theorem, the dispersion relation is:

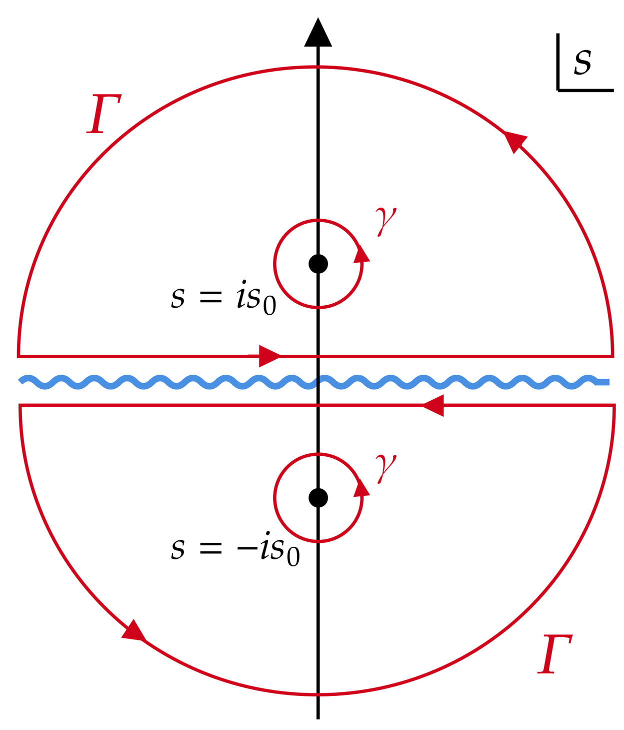

Figure 1: *Contours in the complex s-plane. The blue line represents the branch cut on the real axis.*

$$\Sigma \equiv \oint_\gamma \frac{ds}{2\pi i}\frac{s^3 A(s)}{(s^2+s_0^2)^3} = \frac{2}{\pi}\int_0^\infty ds\,\frac{s^4\sigma(s)}{(s^2+s_0^2)^3} \geq 0\,. \tag{1}$$

where the forward elastic amplitude is obtained by taking the $t \to 0$ limit:

$$A_{ab\to ab}(s) = \mathcal{A}(ab \to ab)|_{t\to 0}\,, \tag{2}$$

Based on scalar-QED EFT operator basis[2], the dispersion relation can be used to derive the positivity bounds.

Scalar QED EFT

We impose $U(1)$ gauge symmetry and use massless scalar ϕ and photon as IR field. The dim-4 Lagrangian is:

$$\mathcal{L}_{[\mathcal{O}]\leq 4} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D^\mu\phi)^\dagger(D_\mu\phi) - \frac{1}{4}\lambda_1(\phi^\dagger\phi)^2\,, \tag{3}$$

The dim-6 operators are:

$$\mathcal{L}_6\cdot\Lambda^2 = c_{\phi^6}(\phi^\dagger\phi)^3 + c_{D^2\phi^4}\phi^\dagger\phi(D_\mu\phi)^\dagger D^\mu\phi + c_{F^2\phi^2}\phi^\dagger\phi F^{\mu\nu}F_{\mu\nu}\,, \tag{4}$$

. The dim-8 operators are:

$$\begin{aligned}\mathcal{L}_8\cdot\Lambda^4 =& c^{(1)}_{F^4}(F_{\mu\nu}F^{\mu\nu})^2 + c^{(2)}_{F^4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2\\ &+ c^{(1)}_{F^2D^2\phi^2}(D_\mu\phi)^\dagger(D_\nu\phi)F^{\mu\rho}F^\nu_{\ \rho} + c^{(2)}_{F^2D^2\phi^2}(D^\mu\phi)^\dagger(D_\mu\phi)F_{\nu\rho}F^{\nu\rho}\\ &+ c^{(1)}_{D^4\phi^4}((D_\mu\phi)^\dagger)^2(D_\nu\phi)^2 + c^{(2)}_{D^4\phi^4}\phi^\dagger\phi(D_\mu D_\nu\phi)^\dagger(D_\mu D_\nu\phi)\\ &+ c_{F^2\phi^4}(\phi^\dagger\phi)^2F_{\mu\nu}F^{\mu\nu} + c_{D^2\phi^6}(\phi^\dagger\phi)^2(D_\mu\phi)^\dagger D^\mu\phi + c_{\phi^8}(\phi^\dagger\phi)^4\,,\end{aligned} \tag{5}$$

Photon-scalar scattering

The tree level positivity bounds are:

$$c^{(1)}_{F^2D^2\phi^2} \leq 0\,. \tag{6}$$

After IR regulation, the subtracted bounds at one-loop are:

$$\begin{aligned}\Sigma'\cdot 2\Lambda^4 \approx& -c^{(1)}_{F^2D^2\phi^2} + \frac{19e^2}{288\pi^2}(2c^{(1)}_{D^4\phi^4} + c^{(2)}_{D^4\phi^4})\\ &- \frac{e^2}{48\pi^2}(2c^{(1)}_{D^4\phi^4} + c^{(2)}_{D^4\phi^4})\log\frac{m^2}{\mu^2} \geq 0\,,\end{aligned} \tag{7}$$

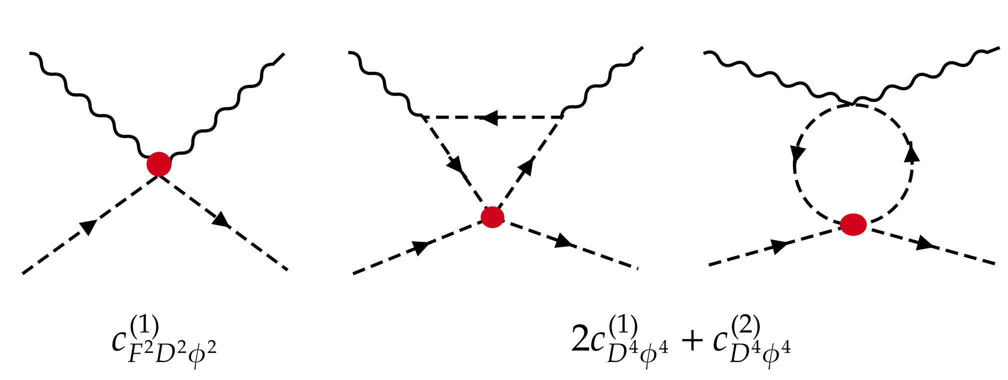

Figure 2: *The corresponding Feynman diagrams for Eq.(7).*

If $c^{(1,2)}_{D^4\phi^4}$ are generated at tree level, Eq.(7) become the tree level bound:

$$2c^{(1)}_{D^4\phi^4} + c^{(2)}_{D^4\phi^4} \geq 0\,. \tag{8}$$

Although Eq.(6) does not necessarily hold,

$$\beta(c^{(1)}_{F^2D^2\phi^2}) = \frac{e^2}{24\pi^2}\left(2c^{(1)}_{D^4\phi^4} + c^{(2)}_{D^4\phi^4}\right)\,, \tag{9}$$

is positive and could restor the tree-level bound. If they are loop generated, Eq.(6) is still valid at LO.

Photon-photon scattering

The tree-level positivity bounds are:

$$c^{(1,2)}_{F^4} \geq 0\,. \tag{10}$$

Consider different linear polarizations:

$$(c^{(1)}_{F^4} - c^{(2)}_{F^4})\cos(2(\alpha_1-\alpha_2)) + c^{(1)}_{F^4} + c^{(2)}_{F^4} \geq 0\,. \tag{11}$$

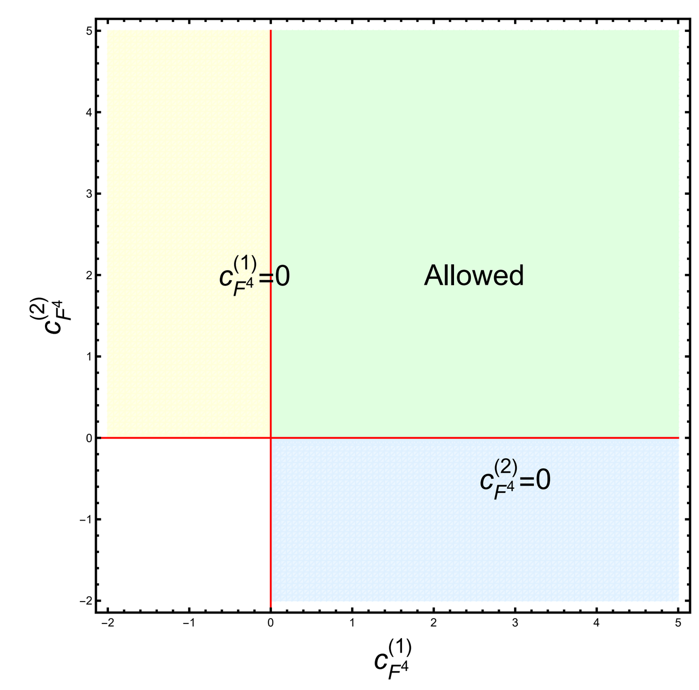

Figure 3: *The positivity bound of $c^{(1)}_{F^4}$ and $c^{(2)}_{F^4}$.*

The allowed region is shown in Fig.3. The subtracted bounds at one-loop are:

$$\begin{aligned}\Sigma'_{yy}\cdot\Lambda^4 =& 16c^{(1)}_{F^4} + \frac{1}{4\pi^2}c^2_{F^2\phi^2}\left(-\frac{3}{2} - 2\log\left(\frac{s_0}{\mu^2}\right) + 4\right)\\ &- \frac{1}{96\pi^2}e^2c^{(1)}_{F^2D^2\phi^2}\left(-\frac{3}{2} - 2\log\left(\frac{s_0}{\mu^2}\right) - 2\log\left(\frac{m_\gamma^2}{\mu^2}\right) + 8\right)\\ &- \frac{1}{144\pi^2}e^2(25c^{(1)}_{F^2D^2\phi^2} + 72c^{(2)}_{F^2D^2\phi^2})\,.\end{aligned} \tag{12}$$

$$\Sigma'_{yx}\cdot\Lambda^4 = 16c^{(2)}_{F^4} + \frac{1}{96\pi^2}e^2c^{(1)}_{F^2D^2\phi^2}\left(\frac{3}{2} + 2\log\left(\frac{s_0}{\mu^2}\right) + 2\log\left(\frac{m_\gamma^2}{\mu^2}\right) - 8\right) - \frac{7}{144\pi^2}e^2c^{(1)}_{F^2D^2\phi^2}\,, \tag{13}$$

Since these operators are generated at one-loop, the LO tree-level results Eq.(10)are still robust.

Top-down perspective

We use the functional matching method to obtain the Wilson coefficients and then check the dispersion relation with UV.

$$\Gamma_{\text{L,EFT}}\left(c_i^{[j]}, \mu = \mu_m\right) = \Gamma_{\text{L,UV}}(g, \mu = \mu_m)\,, \tag{14}$$

UV Model I:

$$\begin{aligned}\mathcal{L}_I =& -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D^\mu\phi)^\dagger(D_\mu\phi) + (D^\mu\Phi)^\dagger(D_\mu\Phi) - M^2\Phi^\dagger\Phi\\ &- \frac{1}{4}\lambda_1(\phi^\dagger\phi)^2 - \frac{1}{4}\lambda_2(\Phi^\dagger\Phi)^2 - \lambda_3\Phi^\dagger\Phi\phi^\dagger\phi - (\frac{1}{2}gM\Phi^\dagger\Phi^\dagger\phi + h.c.)\,.\end{aligned} \tag{15}$$

The dispersion relation is self-consistent with the Matching results.

$$\Sigma' = \frac{2}{\pi}\int_0^\infty ds\,\frac{s^4\sigma(\gamma^+\phi\to\Phi\Phi)}{(s^2+s_0^2)^3} = \frac{g^2e^2}{M^4}\frac{1}{80640\pi^2} + \mathcal{O}\left(\frac{s_0}{M^6}\right)\,, \tag{16}$$

UV Model II:

$$\begin{aligned}\mathcal{L}_{II} =& -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D^\mu\phi)^\dagger(D_\mu\phi) + (D^\mu\Phi)^\dagger(D_\mu\Phi) - M^2\Phi^\dagger\Phi\\ &- \frac{1}{4}\lambda_1(\phi^\dagger\phi)^2 - \frac{1}{4}\lambda_2(\Phi^\dagger\Phi)^2 - \lambda_3\Phi^\dagger\Phi\phi^\dagger\phi - (\frac{1}{2}gM\Phi^\dagger\phi\phi + h.c.)\,.\end{aligned} \tag{17}$$

After small mass m IR regulation, the dispersion relation is self-consistent with the Matching results.

$$\Sigma' = \frac{2}{\pi}\int_0^\infty ds\,\frac{s^4\sigma(\gamma^+\phi\to\Phi\phi^\dagger)}{(s^2+s_0^2)^3} = \frac{e^2g^2}{576M^4\pi^2}\left(11 - 12\log\left(\frac{m^2}{M^2}\right)\right) + \mathcal{O}\left(\frac{s_0}{M^6}\right)\,, \tag{18}$$

Conclusion

- $c^{(1)}_{F^2D^2\phi^2} \leq 0$ does not necessarily hold, its one-loop β-function is subject to a bound that tends to restore the tree-level bound in the IR.
- The tree-level bounds $c^{(1)}_{F^4} \geq 0$ and $c^{(2)}_{F^4} \geq 0$ are valid up to the one-loop level in the UV theory.
- Our results may have important implications on the robustness of experimental tests of positivity bounds.

References

[1]Allan Adams, Nima Arkani-Hamed, Sergei Dubovsky, Alberto Nicolis, and Riccardo Rattazzi. Causality, analyticity and an IR obstruction to UV com- pletion. JHEP, 10:014, 2006

[2] Yunxiao Ye, Xiao Cao, Yu-Hang Wu, and Jiayin Gu. Positivity bounds in scalar-qed eft at one-loop level. Journal of High Energy Physics, 2025(10):1–33, 2025.