Nonlinear Symmetry Fragmentation of Nonabelian Anyons in Symmetry-Enriched Topological Phases

Nianrui Fu, Siyuan Wang, Yu Zhao, Yidun Wan

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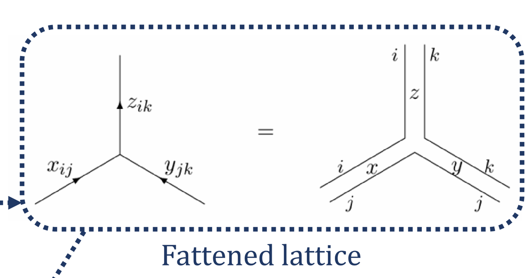
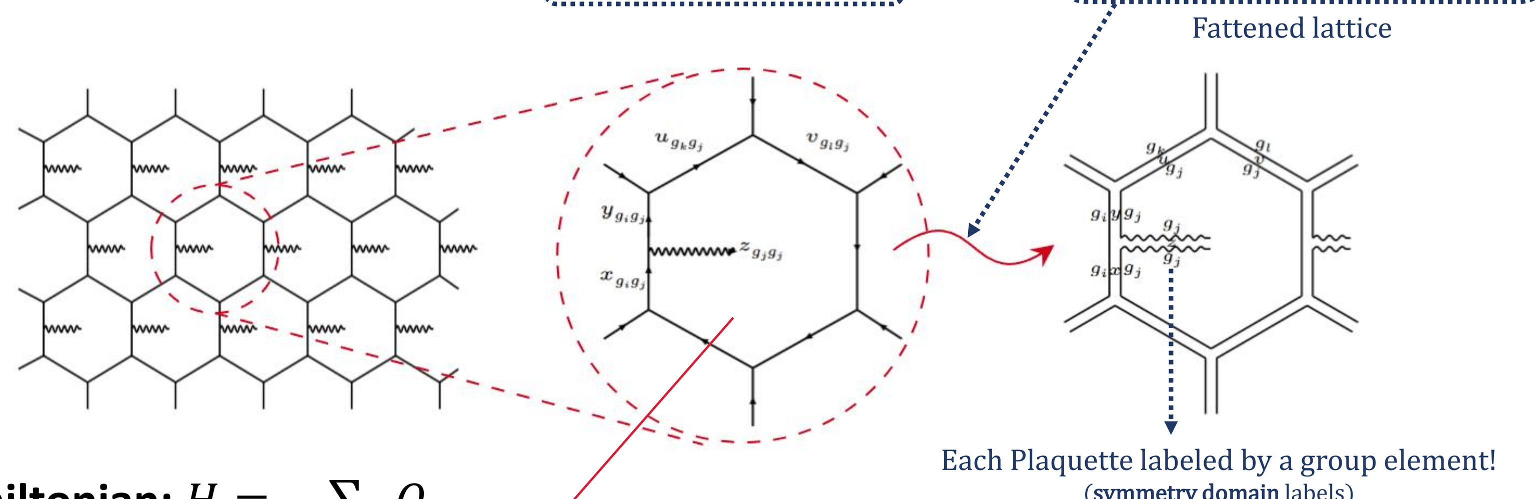
ABSTRACT Symmetry-enriched topological (SET) phases combine intrinsic topological order with global symmetries, giving rise to novel symmetry phenomena. While SET phases with Abelian anyons are relatively well understood, those involving nonabelian anyons remain elusive. This obscurity stems from the multi-dimensional internal gauge spaces intrinsic to nonabelian anyons. We will introduce a systematic framework for constructing exactly solvable lattice models of SET phases based on an enlarged version of the string-net model. Using these explicitly constructed models, we will then focus on how the internal gauge spaces of nonabelian anyons transform under these symmetries. This framework uncovers a universal phenomenon: global symmetry fragmentation (GSF). We will introduce how symmetry-invariant anyons exhibit internal Hilbert space decompositions into eigensubspaces that are labeled by generally fractional symmetry charges. Furthermore, we will see how symmetry-permuted anyons hybridize and fragment their internal spaces in accordance with their symmetry behavior.

Lattice model of SET

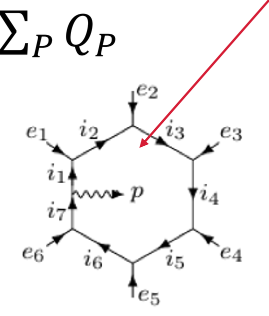
- **Lattice:** honey comb lattice
- **Input:** a multifusion category $\mathcal{M}$
- **Basic d.o.f.s:** simple objects of $\mathcal{M}$ on edges and tails
- **Constraint:** each vertex satisfies fusion rules of $\mathcal{M}$

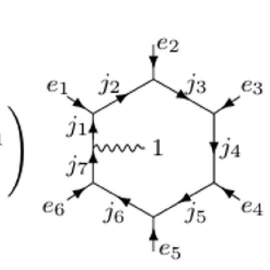




- **Hamiltonian:** $H = -\sum_P Q_P$

$$Q_P := \frac{1}{D}\sum_{s\in L_{\mathscr{F}}} d_s Q_P^s, \quad Q_P^s \text{ [plaquette } p\text{]} := \delta_{p,1}\delta_{j_1,j_7}\sum_{j_k\in L_{\mathscr{F}}}\prod_{k=1}^{6}\left(\sqrt{d_{i_k}d_{j_k}}\,G^{e_k i_k i_{k+1}^*}_{s j_{k+1}^* j_k}\right) \text{ [plaquette } 1\text{]}$$





$[Q_P, Q_{P'}], \forall P, P'$ → Exactly Solvable!

- **Symmetry Action:** $\varphi_g: \mathcal{M}_{g_i,g_j} \to \mathcal{M}_{gg_i,gg_j}$ Transform g_i domain into gg_i domain

Strategies of SET construction

- **SET construction strategies:**

Strategy	Origin of Symmetry	Multifusion Category	Symmetry Action
I	Outer automorphisms of fusion category	$\mathcal{M}_{\mathrm{Out}(\mathscr{F})} = \begin{pmatrix} \mathscr{F}_{g_1g_1} & \mathscr{F}_{g_1g_2} & \cdots \\ \mathscr{F}_{g_2g_1} & \mathscr{F}_{g_2g_2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$	$\varphi_g\left(x_{g_i,g_j}\right) = g(x)_{gg_i,gg_j}$, $\forall g \in \mathrm{Out}(\mathcal{F})$
II	Frobenius algebras and their bimodules	$\mathcal{M} = \begin{pmatrix} {}_{\mathcal{A}_e}\mathrm{Bimod}_{\mathcal{A}_e}(\mathscr{F}) & {}_{\mathcal{A}_e}\mathrm{Bimod}_{\mathcal{A}_{g_i}}(\mathscr{F}) & {}_{\mathcal{A}_e}\mathrm{Bimod}_{\mathcal{A}_{g_j}}(\mathscr{F}) & \cdots \\ {}_{\mathcal{A}_{g_i}}\mathrm{Bimod}_{\mathcal{A}_e}(\mathscr{F}) & {}_{\mathcal{A}_{g_i}}\mathrm{Bimod}_{\mathcal{A}_{g_i}}(\mathscr{F}) & {}_{\mathcal{A}_{g_i}}\mathrm{Bimod}_{\mathcal{A}_{g_j}}(\mathscr{F}) & \cdots \\ {}_{\mathcal{A}_{g_j}}\mathrm{Bimod}_{\mathcal{A}_e}(\mathscr{F}) & {}_{\mathcal{A}_{g_j}}\mathrm{Bimod}_{\mathcal{A}_{g_i}}(\mathscr{F}) & {}_{\mathcal{A}_{g_j}}\mathrm{Bimod}_{\mathcal{A}_{g_j}}(\mathscr{F}) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$	Isomorphism between bimodules
III	Externally imposed symmetry G (independent of category)	$\mathcal{M} = \begin{pmatrix} \mathscr{F}_{g_1g_1} & \mathscr{F}_{g_1g_2} & \cdots & \mathscr{F}_{g_1g_n} \\ \mathscr{F}_{g_2g_1} & \mathscr{F}_{g_2g_2} & \cdots & \mathscr{F}_{g_2g_n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathscr{F}_{g_ng_1} & \mathscr{F}_{g_ng_2} & \cdots & \mathscr{F}_{g_ng_n} \end{pmatrix}$	• Diagonal elements: trivial • Off-diagonal elements: free choices

- **Gauging strategies:** identify states related by symmetry actions
- **Examples:**

	Strategy I		Strategy II			Strategy III		
Original Topological order	$D(\mathrm{Vec}(\mathbb{Z}_3))$	$D(\mathrm{Vec}(\mathbb{Z}_2\times\mathbb{Z}_2))$	$D(\mathrm{Vec}(\mathbb{Z}_2))$	$D(\mathrm{Vec}(\mathbb{Z}_2\times\mathbb{Z}_2))$	$D(\mathrm{Vec}(S_3))$	$D(\mathrm{Vec}(\mathbb{Z}_2))$		Trivial Phase
Enriched by Symmetry	Charge-conjugation symmetry	S_3 symmetry	EM-exchange symmetry	$\mathbb{Z}_2$ symmetry	EM-exchange symmetry	Imposed $\mathbb{Z}_2$ symmetry		Imposed symmetry G
Parent Phase (after gauging)	$D(\mathrm{Vec}(S_3))$	$D(\mathrm{Vec}(S_4))$	$D(\mathrm{Ising})$	$D(\mathrm{Rep}(D_4))$	$D(A_F^5)$	$D(\mathrm{Vec}(\mathbb{Z}_2\times\mathbb{Z}_2))$	$D(\mathrm{Vec}(\mathbb{Z}_4))$	$D(\mathrm{Rep}(G))$

First nonabelian symmetry enriched topological order!

Different choices of symmetry action

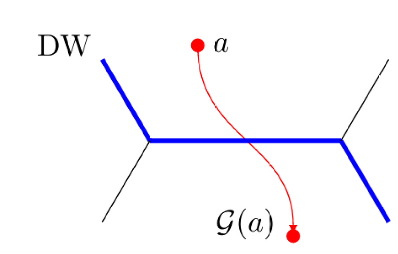
SPT!

Blood Symmetry: emerge from the intrinsic property of fusion category itself

Adopted Symmetry: externally imposed

Nonlinear symmetry fragmentation

- **Cross domain wall mechanism:**



cross domain wall = symmetry transformation

- **Vec(S_3) Case (EM exchange symmetry):**

Vec(S_3) anyon

Anyon	A	B	C	D	E	F	G	H
Flux	1	1	1	s, rs, sr	s, rs, sr	r, r^2	r, r^2	r, r^2
Charge	1	1	1, 2	1	1	1	1	1

Corss domain wall mechanism →

Anyon Internal Basis	EM-exchange transformed basis
A	A
B	B
C_{1_1}	F_r
C_{1_2}	F_{r^2}
F_r	C_{1_1}
F_{r^2}	C_{1_2}
G_r	$e^{i\frac{2\pi}{3}}G_r$
G_{r^2}	$e^{i\frac{2\pi}{3}}G_{r^2}$
H_r	$e^{-i\frac{2\pi}{3}}H_{r^2}$
H_{r^2}	$e^{-i\frac{2\pi}{3}}H_r$
D_s	$\frac{1}{\sqrt{3}}(D_s + D_{rs} + D_{sr})$
D_{rs}	$\frac{1}{\sqrt{3}}(D_s + e^{i\frac{2\pi}{3}}D_{rs} + e^{-i\frac{2\pi}{3}}D_{sr})$
D_{sr}	$\frac{1}{\sqrt{3}}(D_s + e^{-i\frac{2\pi}{3}}D_{rs} + e^{i\frac{2\pi}{3}}D_{sr})$
E_s	$\frac{1}{\sqrt{3}}(E_s + E_{rs} + E_{sr})$
E_{rs}	$\frac{1}{\sqrt{3}}(E_s + e^{i\frac{2\pi}{3}}E_{rs} + e^{-i\frac{2\pi}{3}}E_{sr})$
E_{sr}	$\frac{1}{\sqrt{3}}(E_s + e^{-i\frac{2\pi}{3}}E_{rs} + e^{i\frac{2\pi}{3}}E_{sr})$

Symmetry Tansformation Pattern of Vec(S_3) Anyon

- **Fragmentation Phenomena:** the internal spaces of nonabelian anyons can fragment into eigensubspaces carrying fractional charges.

Non-Abelian anyons	Definite symmetry charge states (spaces)	Symmetry charge
$C \oplus F$	$\mathrm{span}\{C_{1_1}+F_r,\ C_{1_2}+F_{r^2}\}$	0
	$\mathrm{span}\{C_{1_1}-F_r,\ C_{1_2}-F_{r^2}\}$	$\frac{1}{2}$
G	$\mathrm{span}\{G_r,\ G_{r^2}\}$	$\frac{1}{3}$
H	$H_r + H_{r^2}$	$\frac{2}{3}$
	$H_r - H_{r^2}$	$\frac{1}{6}$
D	$D_{rs} - D_{sr}$	$\frac{1}{4}$
	$(1+\sqrt{3})D_s + D_{rs} + D_{sr}$	0
	$(1-\sqrt{3})D_s + D_{rs} + D_{sr}$	$\frac{1}{2}$
E	$E_{rs} - E_{sr}$	$\frac{1}{4}$
	$(1+\sqrt{3})E_s + E_{rs} + E_{sr}$	0
	$(1-\sqrt{3})E_s + E_{rs} + E_{sr}$	$\frac{1}{2}$

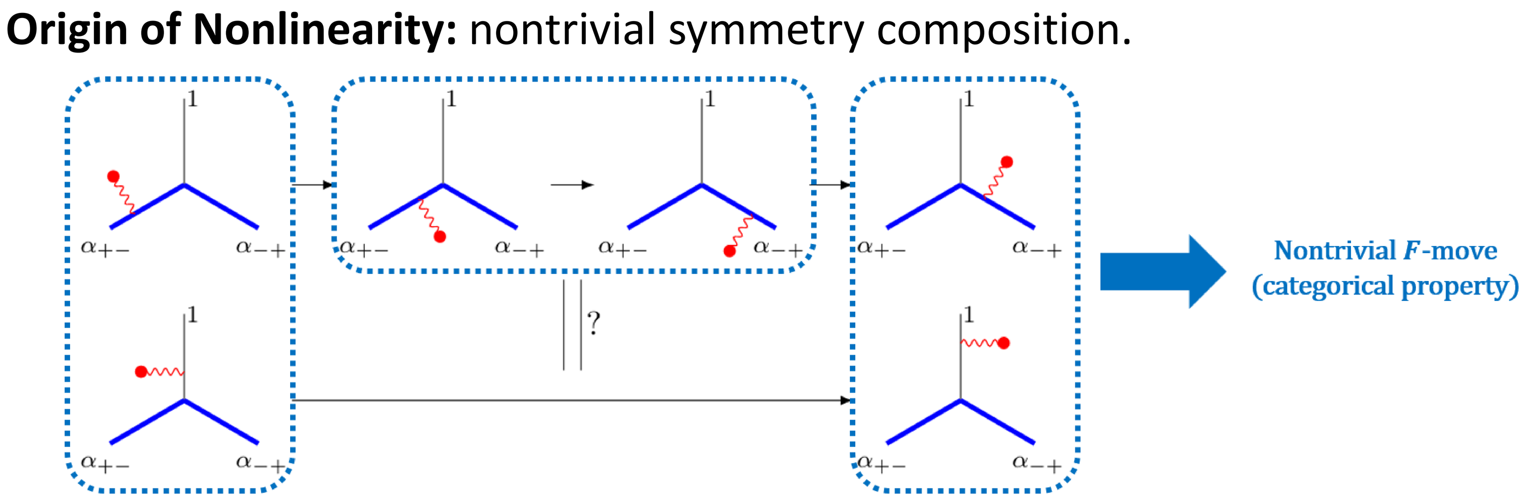
Nonlinearity!

- **Nonlinearity:** $\sum_b \omega_{ab}\rho^{\tilde{J}}_{ab}(\mathcal{G}_{em})\rho^{\tilde{J}}_{bc}(\mathcal{G}_{em}) = \rho^{\tilde{J}}_{ac}(1) = \mathbb{1}.$ (ω_{ab}: Rank-2 tensor)

Example: Not projective representations!

$\rho^{(C\oplus F)_\pm}(\mathcal{G}_{em})\rho^{(C\oplus F)_\pm}(\mathcal{G}_{em}) = \rho^{(C\oplus F)_\pm}(1)$ $(V_{(C\oplus F)_\pm} = \mathrm{span}\{C_{1_1}\pm F_r,\ C_{1_2}\pm F_{r^2}\})$	$\rho^{G_r}(\mathcal{G}_{em})\rho^{G_r}(\mathcal{G}_{em}) = e^{-i\frac{2\pi}{3}}\rho^{G_r}(1)$ $\rho^{G_{r^2}}(\mathcal{G}_{em})\rho^{G_{r^2}}(\mathcal{G}_{em}) = e^{-i\frac{2\pi}{3}}\rho^{G_{r^2}}(1)$	$\rho^{H_+}(\mathcal{G}_{em})\rho^{H_+}(\mathcal{G}_{em}) = e^{i\frac{2\pi}{3}}\rho^{H_+}(1)$ $\rho^{H_-}(\mathcal{G}_{em})\rho^{H_-}(\mathcal{G}_{em}) = e^{i\frac{2\pi}{3}}\rho^{H_-}(1)$ $(H_+ = H_r + H_{r^2}, H_- = H_r - H_{r^2})$

- **Origin of Nonlinearity:** nontrivial symmetry composition.



Symmetry-gauging family

- **Symmetry-gauging family:** Topological phases related by gauging their blood symmetries. Could be depicted by an oriented web:

$D(\mathtt{Vec}(\mathbb{Z}_3)) \xrightarrow{s} D(\mathtt{Vec}(S_3)) \xrightarrow{\alpha=\sigma,\ \beta=s\sigma} D(A_5^F)$

$D(\mathtt{Vec}(\mathbb{Z}_3)) \xrightarrow{\sigma} D(\mathbb{Z}_3\text{−}\mathtt{Ising}) \xrightarrow{s,\ \sigma s} D(A_5^F)$