Finite-Size Scaling of the Full Eigenstate Thermalization in Quantum Spin Chains

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Abstract

Despite the unitary evolution of closed quantum systems, long-time expectations of local observables are well described by thermal ensembles, providing the foundation of quantum statistical mechanics. A promising route to understanding this quantum thermalization is the eigenstate thermalization hypothesis (ETH). Subsequent studies have extended this concept to the full ETH, which captures higher-order correlations among matrix elements through nontrivial relations. In this work, we perform a detailed exact-diagonalization study of finite-size corrections to these relations in the canonical ensemble. We distinguish two distinct sources of corrections: those arising from energy fluctuations, which decay polynomially with system size, and those originating from fluctuations within each energy window, which decay exponentially with system size. In particular, our analysis resolves the puzzle that, for certain observables, finite-size corrections exhibit anomalous growth with increasing system size even in chaotic systems. Our results provide a systematic and practical methodology for validating the full ETH in quantum many-body systems.

1. Background

Traditional **ETH** posits that individual many-body eigenstates encode thermal behavior for local observables:

$$O_{ij} = O(E_i)\delta_{ij} + e^{-S(E^+)/2} F^{(2)}_{E^+}(\omega_{ij}) R_{ij}$$

where $E^+ = (E_i + E_j)/2$ and $\omega_{ij} = E_i - E_j$. Standard ETH treats R_{ij} as a Gaussian random variable, which fails to capture nontrivial higher-order correlation functions.
To overcome this limitation, the **Full ETH**(FETH) was introduced:

$$\overline{O_{i_1 i_2} O_{i_2 i_3} \cdots O_{i_q i_1}} = e^{-(q-1)S(E^+)} F^{(q)}_{E^+}(\omega)$$

$$\overline{O_{i_1 i_2} \cdots O_{i_{k-1} i_1} O_{i_1 i_{k+}} \cdots O_{i_q i_1}} = \overline{O_{i_1 i_2} \cdots O_{i_{k-1} i_1}}\ \overline{O_{i_1 i_{k+1}} \cdots O_{i_q i_1}}$$

We define the q-point correlation function:

$$C^{(q)}(t_1, t_2, \cdots, t_{q-1}) \equiv \langle O(t_1) \cdots O(t_{q-1}) O(0) \rangle,$$

which can be expressed by the free cumulant under the FETH assumption:

$$\langle A(t_1)A(t_2)\dots A(t_n)\rangle_\beta = \sum_{\pi \in \mathrm{NC}(n)} \kappa^\beta_\pi (A(t_1)A(t_2)\dots A(t_n))$$

$$\kappa^\beta_\pi (A(t_1)A(t_2)\dots A(t_n)) = \prod_{b\in\pi} \kappa^\beta_{|b|}\Big(\prod_{j\in b} A(t_j)\Big)$$

$$k^{\mathrm{ETH}}_n (A(t_1)A(t_2)\dots A(t_n)) = \mathrm{FT}\Big[F^{(n)}_{e_\beta}(\vec{\omega}) e^{-\beta\vec{\omega}\cdot\vec{\ell}_n}\Big]$$

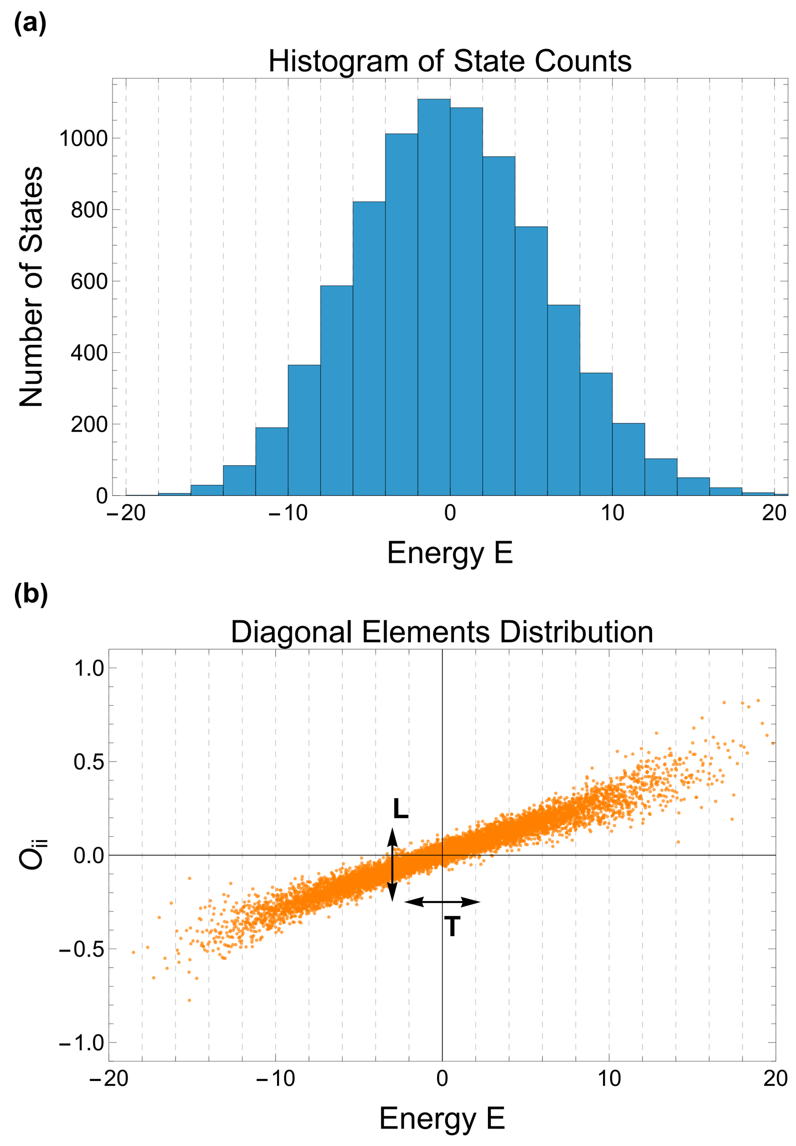
2. Decomposition Strategy

We focus on finite-size corrections to correlation functions in the canonical ensemble. Taking the two-point function error F_{11} as a key benchmark:

$$F_{11} \equiv C^{(2)}_{ETH}(t) - C^{(2)}(t) = \frac{1}{D}\sum_i O_{ii}^2 - \left(\frac{1}{D}\sum_i O_{ii}\right)^2$$

We propose a decomposition into longitudinal (F^L_{11}) and transverse (F^T_{11}) components, $F_{11} = F^L_{11} + F^T_{11}$:

- **Longitudinal Variance (F^L_{11}):** Measures the variance of O_{ii} within each narrow energy window. It decays **exponentially** with system size: $F^L_{11} \propto e^{-bL}$.
- **Transverse Variance (F^T_{11}):** Measures the variance between different energy windows. It can be computed via thermal ensembles and decays **polynomially** (power-law): $F^T_{11} \propto L^{-1}$.



The explicit definitions for the decomposed longitudinal (F^{L}_{11}) and transverse (F^{T}_{11}) variances are given by:

$$F^{\mathrm{L}}_{11} \equiv \sum_E P_E \left[\frac{1}{d_E}\sum_{i\in E} O_{ii}^2 - \left(\frac{1}{d_E}\sum_{i\in E} O_{ii}\right)^2\right],$$

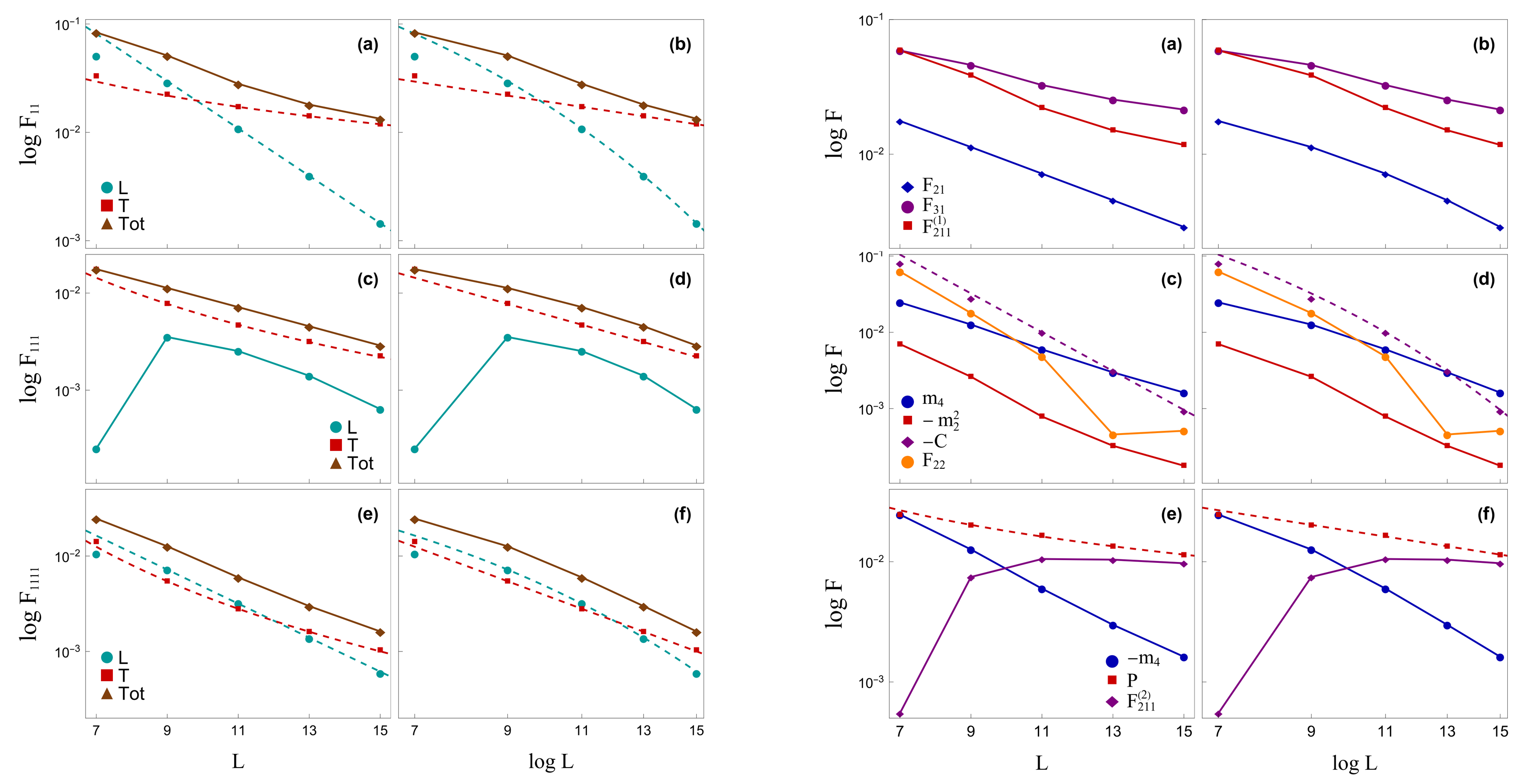
$$F^{\mathrm{T}}_{11} \equiv \sum_E P_E \left[\left(\frac{1}{d_E}\sum_{i\in E} O_{ii}\right)^2\right] - \left[\sum_E P_E \left(\frac{1}{d_E}\sum_{i\in E} O_{ii}\right)\right]^2.$$

3. Numerical Results

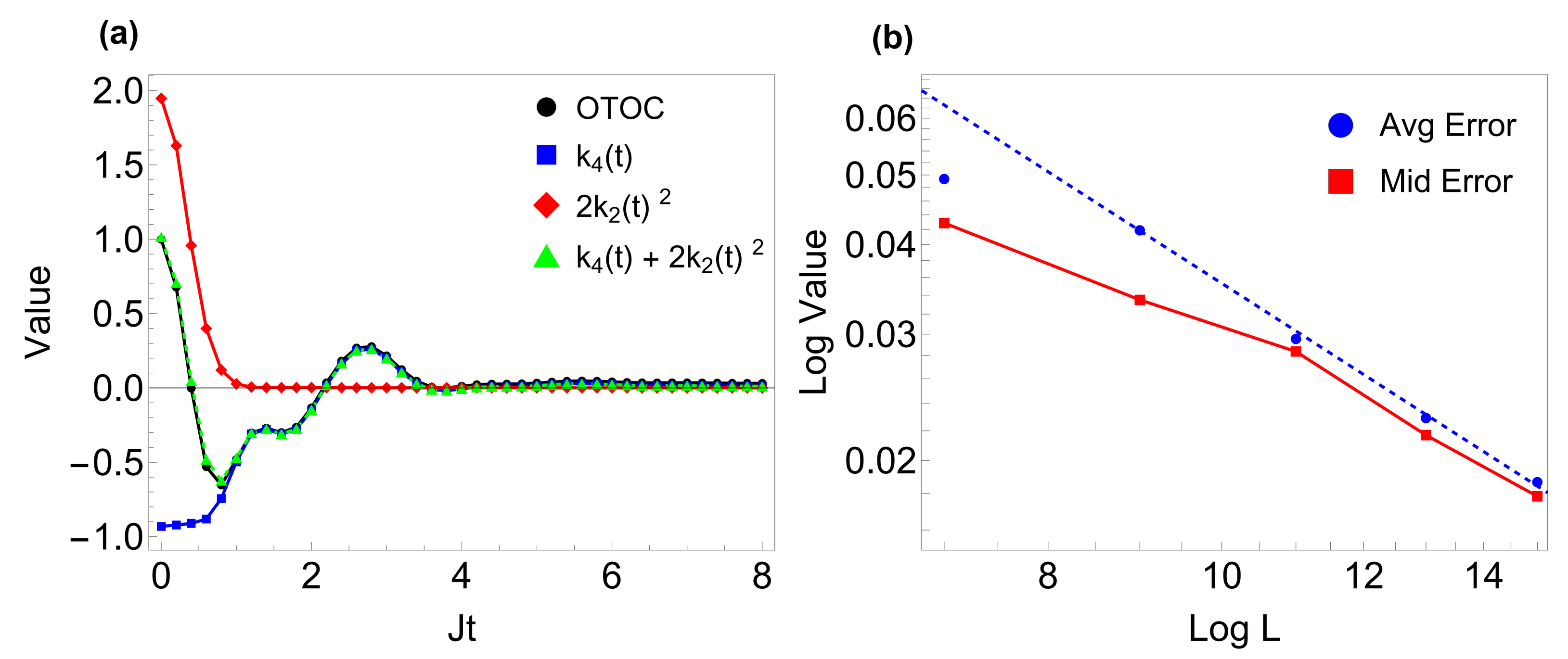
We perform **Exact Diagonalization (ED)** to study the finite-size scaling of the FET relations.

Example 1: Mixed-field Ising model ($J = 1$, $w = 1.05$, and $h = \frac{\sqrt{5}-1}{2}$):

$$\hat{H} = J\sum_{i=1}^{L} Z_i Z_{i+1} + w\sum_{i=1}^{L} X_i + h\sum_{i=1}^{L} Z_i$$

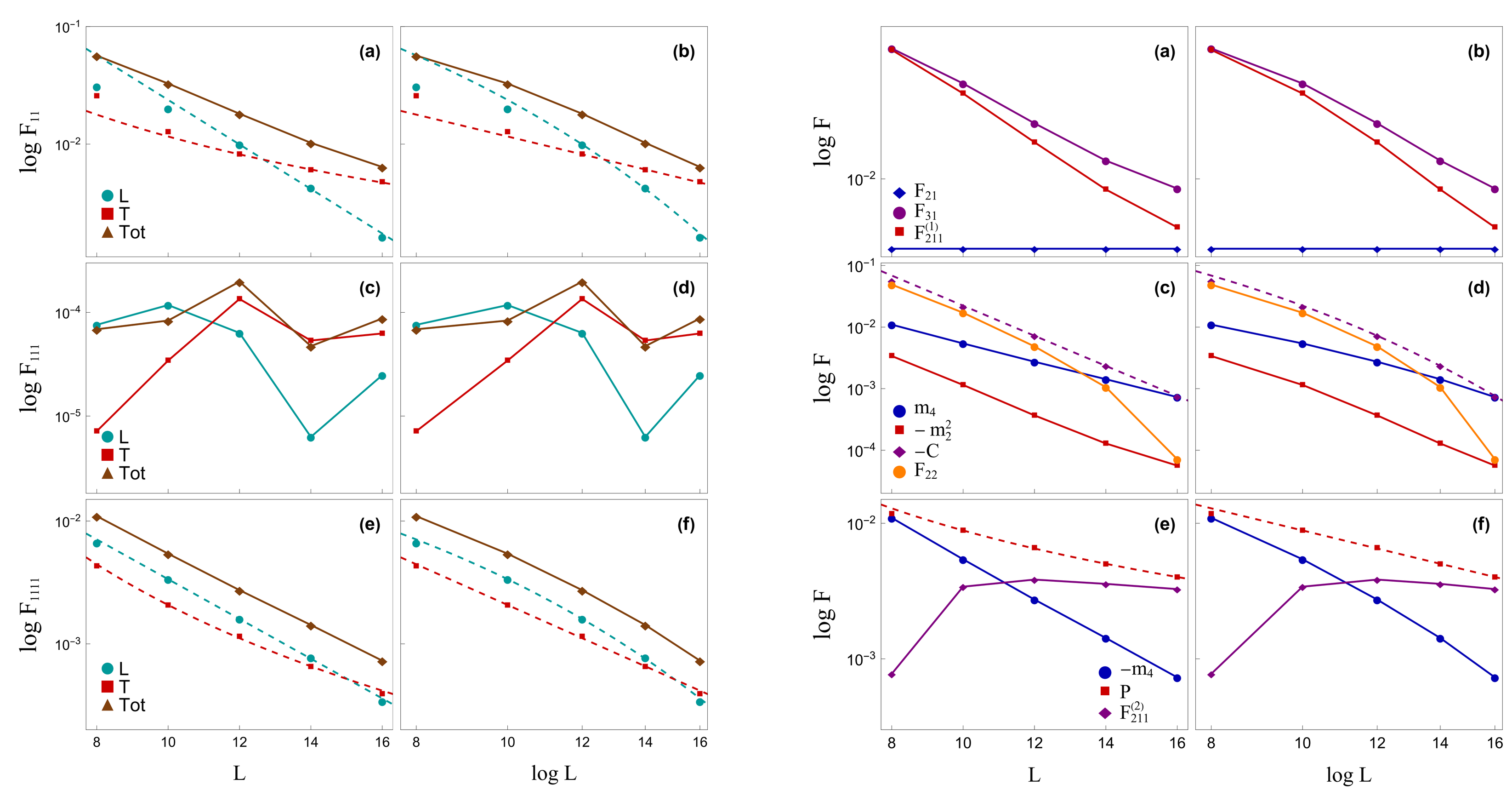


Finite-size scaling at **finite time:** $\mathrm{OTOC}(t) \approx k_4(t,0,t) + 2k_2(t)^2 \equiv \mathrm{OTOC}_{\mathrm{ETH}}(t)$.



Example 2: Random-field XXZ model ($J = 1$, $J_z = 1.05$, and $h_i \in [-0.75, 0.75]$):

$$\hat{H} = J\sum_{i=1}^{L}(X_i X_{i+1} + Y_i Y_{i+1}) + J_z\sum_{i=1}^{L} Z_i Z_{i+1} + \sum_{i=1}^{L} h_i Z_i$$



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