Polarization of fermions at local thermodynamic equilibrium

F. B., to appear soon

Revisiting

F. B., V. Chandra, L. Del Zanna, E. Grossi, Ann. Phys. 338, 32 (2013),

OUTLINE

- Polarization: single particle vs Quantum Field Theory
- Covariant Wigner function
- Angular momentum operator
- Local and global thermodynamic equilibrium

Spin in relativistic theories

Definition of spin operators:

$$\widehat{S}^\mu = -\frac{1}{2m}\epsilon^{\mu\nu\rho\sigma}\widehat{J}_{\nu\rho}\widehat{P}_\sigma$$

Since it is orthogonal to four-momentum:

$$\widehat{S}(p) = \sum_{i=1}^{3} \widehat{S}_i(p) n_i(p)$$

$$n_i(p) = [p]e_i$$

[p] = standard Lorentz transformation

$$\widehat{P}|p,\sigma\rangle = p|p,\sigma\rangle \qquad \text{and} \qquad \widehat{S}_3(p)|p,\sigma\rangle = \sigma|p,\sigma\rangle$$

Eigenstates of massive particles

$$\widehat{\Lambda}|p,\sigma\rangle = \sum_\tau |\Lambda p,\tau\rangle D^S_{\tau\sigma}([\Lambda p]^{-1}\Lambda[p])$$

Within round brackets: Wigner rotation

Definition of *mean spin vector* for a single relativistic particle

$$S^\mu(p) = \sum_\sigma \langle p, \sigma | \widehat{S}^\mu \widehat{\rho} | p, \sigma \rangle = \mathrm{tr}(\widehat{S}^\mu \widehat{\rho})$$

$$\Theta(p)_{\sigma\tau} \equiv \langle p, \sigma | \widehat{\rho} | p, \tau \rangle$$

Spin density matrix

$$S^\mu(p) = \sum_{i=1}^{3} [p]^\mu_i \, \mathrm{tr}(D^S(\mathsf{J}^i)\Theta(p)),$$

In covariant form:

$$S^\mu(p) = -\frac{1}{2}\epsilon^{\alpha\lambda\nu\rho}\hat{t}_\rho [p]^\mu_\alpha \, \mathrm{tr}(D^S(J_{\lambda\nu})\Theta(p))$$

$$S^\mu(p) = -\frac{1}{2m}\epsilon^{\mu\beta\gamma\delta} p_\delta \, \mathrm{tr}(D^S([p]^{-1}) D^S(J_{\beta\gamma}) D^S([p])\Theta(p))$$

Quantum Field Theory?

There is no single particle density operator? How to define the spin density matrix?

$$\Theta(p)_{\sigma\tau} = \frac{\mathrm{Tr}(\widehat{\rho}\,\widehat{a}_\tau^\dagger(p)\widehat{a}_\sigma(p))}{\sum_\sigma \mathrm{Tr}(\widehat{\rho}\,\widehat{a}_\sigma^\dagger(p)\widehat{a}_\sigma(p))},$$

The density operator $\widehat{\rho}$ now is the state of the field and no longer the state of a single particle

The Wigner function

We need something local because we will have to deal with local thermodynamic equilibrium

Definition for a scalar field

$$\widehat{W}(x,k) = \frac{2}{(2\pi)^4} \int \mathrm{d}^4 y \ : \widehat{\psi}^\dagger(x+y/2)\widehat{\psi}(x-y/2) : \mathrm{e}^{-iy\cdot k}$$

$$W(x,k) = \mathrm{Tr}(\widehat{\rho}\widehat{W}(x,k))$$

Decomposition in time and space-like parts

$$W(x,k) = W(x,k)\theta(k^2)\theta(k^0) + W(x,k)\theta(k^2)\theta(-k^0) + W(x,k)\theta(-k^2) \equiv W(x,k)_+ + W(x,k)_- + W(x,k)_S$$

An expression of W(x,k) can be obtained by plugging the free field solution in terms of plane waves

$$\widehat{\psi}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3 \mathrm{p}}{2\varepsilon} \ \mathrm{e}^{-ip\cdot x}\widehat{a}(p) + \mathrm{e}^{ip\cdot x}\widehat{b}^\dagger(p)$$

Example: current

$$\langle \ \rangle = \mathrm{Tr}(\widehat{\rho} \ \ )$$

$$j^\mu(x) = i\langle : \widehat{\psi}^\dagger(x)\overleftrightarrow{\partial^\mu}\widehat{\psi}(x) : \rangle = \int \mathrm{d}^4k \; k^\mu W(x,k)$$

Plugging the free field expansion, it turns out that the current can be written

$$j^\mu(x) = \int \mathrm{d}^4k \; k^\mu W(x,k) = \mathrm{Re} \int \frac{\mathrm{d}^3\mathrm{p}}{2\varepsilon} p^\mu \left[f_c(x,p) - \bar{f}_c(x,p) + g_c(x,p) \right]$$

where *fc* is a complex phase space distribution function including quantum corrections

$$f_c(x,p) = \frac{1}{(2\pi)^3} \int \frac{\mathrm{d}^3\mathrm{p}'}{2\varepsilon'} \; \mathrm{e}^{i(p-p')\cdot x} \langle \widehat{a}^\dagger(p)\widehat{a}(p') \rangle$$

It is the generalization of the classical phase space distribution function and it reproduces BE distribution in (familiar) global thermodynamic equilibrium

The Wigner function of the Dirac field

It is a 4 x 4 matrix

$$\widehat{W}(x,k)_{AB} = -\frac{1}{(2\pi)^4}\int \mathrm{d}^4 y \, \mathrm{e}^{-ik\cdot y} : \Psi_A(x-y/2)\overline{\Psi}_B(x+y/2) :$$
$$= \frac{1}{(2\pi)^4}\int \mathrm{d}^4 y \, \mathrm{e}^{-ik\cdot y} : \overline{\Psi}_B(x+y/2)\Psi_A(x-y/2) :$$

$$\Psi(x)_A = \sum_\sigma \frac{1}{(2\pi)^{3/2}}\int \frac{\mathrm{d}^3\mathrm{p}}{2\varepsilon} \, \widehat{a}_\sigma(p)u_\sigma(p)_A e^{-ip\cdot x} + \widehat{b}^\dagger_\sigma(p)v_\sigma(p)_A e^{ip\cdot x}$$

$$W(x,k) = W(x,k)\theta(k^2)\theta(k^0) + W(x,k)\theta(k^2)\theta(-k^0) + W(x,k)\theta(-k^2) \equiv W(x,k)_+ + W(x,k)_- + W(x,k)_S$$

Example: current

$$j^\mu(x) = \langle : \overline{\Psi}(x)\gamma^\mu \Psi : \rangle = \int \mathrm{d}^4 k \, \mathrm{tr}(\gamma^\mu W(x,k))$$

$$j^\mu_+(x) = \int \mathrm{d}^4 k \, \mathrm{tr}(\gamma^\mu W(x,k)_+) = \sum_{\sigma,\tau} \frac{1}{(2\pi)^3} \int \frac{\mathrm{d}^3 \mathrm{p}}{2\varepsilon} \frac{\mathrm{d}^3 \mathrm{p}'}{2\varepsilon'} \mathrm{e}^{-i(p-p')\cdot x} \langle \widehat{a}^\dagger_\sigma(p) \widehat{a}_\tau(p') \rangle \bar{u}_\tau(p') \gamma^\mu u_\sigma(p)$$

Unlike the scalar field, it cannot be put in the form of the integral of a single function *f(x,p)*

For free field, the Wigner function fulfills:

$$k^\mu \partial_\mu W_\pm(x,k) = k^\mu \partial_\mu W_S(x,k) = 0$$

$$\int_\Sigma \mathrm{d}\Sigma_\mu k^\mu W(x,k)$$

is independent of the 3D hypersurface Σ

Moreover, it can be shown that after integration k becomes on-shell!

On-shellness

$$\frac{1}{2\varepsilon_k}\delta(k^0 - \varepsilon_k)\tilde{W}_+(k) = \int \mathrm{d}\Sigma_\mu k^\mu W_+(x,k)$$

$$\frac{1}{2\varepsilon_k}\delta(k^0 + \varepsilon_k)\tilde{W}_-(k) = \int \mathrm{d}\Sigma_\mu k^\mu W_-(x,k)$$

$$\tilde{W}_+(k) = \sum_{\sigma,\tau} \frac{1}{2}\langle \widehat{a}^\dagger_\tau(k)\widehat{a}_\sigma(k)\rangle u_\sigma(k)\bar{u}_\tau(k)$$

$$(\not{k} - m)\tilde{W}_+(k) = \tilde{W}_+(k)(\not{k} - m) = 0$$

$$\bar{u}_\sigma(k)\tilde{W}_+(k)u_\tau(k) = 2m^2\langle \widehat{a}^\dagger_\tau(k)\widehat{a}_\sigma(k)\rangle$$

Eventually, particle spectrum can be written in terms of the Wigner function!

$$\varepsilon\frac{\mathrm{d}N}{\mathrm{d}^3\mathrm{p}} = \frac{1}{2m}\mathrm{tr}_4\tilde{W}_+(k)$$

Spin vector and Wigner function

Keeping in mind:

$$\bar{u}_\sigma(k)\tilde{W}_+(k)u_\tau(k) = 2m^2\langle\widehat{a}_\tau^\dagger(k)\widehat{a}_\sigma(k)\rangle$$

And the definition of the spin density matrix:

$$\Theta(p)_{\sigma\tau} = \frac{\mathrm{Tr}(\widehat{\rho}\,\widehat{a}_\tau^\dagger(p)\widehat{a}_\sigma(p))}{\sum_\sigma \mathrm{Tr}(\widehat{\rho}\,\widehat{a}_\sigma^\dagger(p)\widehat{a}_\sigma(p))},$$

We can write it in terms of the Wigner function with on-shell arguments

$$\Theta(p)_{\sigma\tau} = \frac{\bar{u}_\sigma(p)\tilde{W}_+(p)u_\tau(p)}{\sum_\sigma \bar{u}_\sigma(p)\tilde{W}_+(p)u_\tau(p)}$$

$$\Theta(p)_{\sigma\tau} = \frac{\int_\Sigma \mathrm{d}\Sigma_\mu p^\mu \bar{u}_\sigma(p)W_+(x,p)u_\tau(p)}{\sum_\sigma \int_\Sigma \mathrm{d}\Sigma_\mu p^\mu \bar{u}_\sigma(p)W_+(x,p)u_\tau(p)}$$

Integration can be done on an arbitrary 3D hypersurface, but there is a most convenient one….

Back to the spin vector

General expression for a particle with spin S

$$S^\mu(p) = -\frac{1}{2m}\epsilon^{\mu\beta\gamma\delta}p_\delta \mathrm{tr}(D^S([p]^{-1})D^S(J_{\beta\gamma})D^S([p])\Theta(p))$$

$$U(p) = \sqrt{m}\begin{pmatrix} D^S([p]) \\ D^S([p]^{\dagger -1}) \end{pmatrix} \qquad V(p) = \sqrt{m}\begin{pmatrix} D^S([p]C^{-1}) \\ D^S([p]^{\dagger -1}C) \end{pmatrix}$$

Specializing to the Dirac field:

$$U(p)\bar{U}(p) = \sum_\sigma u_\sigma(p)\bar{u}_\sigma(p) = \not{p} + m$$

After some manipulations, one obtains:

$$S^\mu(p) = -\frac{1}{2}\epsilon^{\mu\beta\gamma\delta}p_\delta \frac{\int_\Sigma \mathrm{d}\Sigma_\mu \mathrm{tr}_4(\Sigma_{\beta\gamma}\gamma^\mu W_+(x,p))}{\int_\Sigma \mathrm{d}\Sigma_\mu p^\mu \mathrm{tr}_4 W_+(x,p)} = -\frac{1}{4}\epsilon^{\mu\beta\gamma\delta}p_\delta \frac{\int_\Sigma \mathrm{d}\Sigma_\mu \mathrm{tr}_4(\{\gamma^\mu, \Sigma_{\beta\gamma}\}W_+(x,p))}{\int_\Sigma \mathrm{d}\Sigma_\mu p^\mu \mathrm{tr}_4 W_+(x,p)}$$

Polarization from total angular momentum

Starting from a different definition of the mean spin vector: divide the *total* angular momentum of the particles with momentum *p* by their number

$$S^\mu(p) = -\frac{1}{2m}\epsilon^{\mu\nu\rho\sigma}p_\sigma \frac{\tilde{J}_{\nu\rho}(p)}{n(p)}$$

Does $\tilde{J}_{\nu\rho}(p)$ exist?

Generally conserved charge is an integral of a divergence-less current, which, in turn can be written as an integral of a functional of *W(x,k)*

$$Q^{\mu_1\dots\mu_N} = \int_\Sigma \mathrm{d}\Sigma_\lambda \; J^{\lambda\mu_1\dots\mu_N}$$

$$J^{\lambda\mu_1\dots\mu_N} = \int \mathrm{d}^4k \; \mathcal{F}[W(x,k)]^{\lambda\mu_1\dots\mu_N}$$

Example: charge current

$$j^\mu(x) = \langle : \overline{\Psi}(x)\gamma^\mu \Psi : \rangle = \int \mathrm{d}^4k \; \mathrm{tr}(\gamma^\mu W(x,k))$$

Hence:

$$Q^{\mu_1\ldots\mu_N} = \int_\Sigma \mathrm{d}\Sigma_\lambda J^{\lambda\mu_1\ldots\mu_N} = \int_\Sigma \mathrm{d}\Sigma_\lambda \int \mathrm{d}^4k\, \mathcal{F}[W(x,k)]^{\lambda\mu_1\ldots\mu_N}$$
$$= \int_{t=0} \mathrm{d}^3\mathrm{x} \int \mathrm{d}^4k\, \mathcal{F}[W(x,k)]^{0\mu_1\ldots\mu_N} = \int \mathrm{d}^4k \left(\int_{t=0} \mathrm{d}^3\mathrm{x}\, \mathcal{F}[W(x,k)]^{0\mu_1\ldots\mu_N} \right)$$

For free fields, the integration in x implies that the resulting k is on-shell !

$$Q^{\mu_1\ldots\mu_N} = \int \mathrm{d}^4k\, \delta(k^2-m^2) Q(k)^{\mu_1\ldots\mu_N} = \int \mathrm{d}^3\mathrm{k} \left(\int \mathrm{d}k^0\, \delta(k^2-m^2) Q(k)^{\mu_1\ldots\mu_N} \right)$$
$$= \int \mathrm{d}^3\mathrm{k}\, \tilde{Q}_+(k)^{\mu_1\ldots\mu_N} + \tilde{Q}_-(k)^{\mu_1\ldots\mu_N}$$

$$\tilde{Q}_+(k)^{\mu_1\ldots\mu_N} = \int \mathrm{d}k^0 \int \mathrm{d}\Sigma_\lambda\, \mathcal{F}[W_+(x,k)]^{\lambda\mu_1\ldots\mu_N}$$

These functions are INDEPENDENT of the so-called pseudo-gauge transformations of the currents

$$j^\mu \to j^\mu + \partial_\lambda A^{\lambda\mu}$$

Angular momentum

$$\mathcal{J}^{\lambda,\mu\nu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu} + \mathcal{S}^{\lambda,\mu\nu}$$

Canonical tensors

$$T^{\mu\nu}(x) = \frac{i}{2}\langle : \overline{\Psi}(x)\gamma^\mu \overleftrightarrow{\partial}^\nu \Psi(x) : \rangle = \int \mathrm{d}^4k \; k^\nu \mathrm{tr}_4(\gamma^\mu W(x,k))$$

$$\mathcal{S}^{\lambda,\mu\nu} = \frac{1}{2}\langle : \overline{\Psi}(x)\{\gamma^\lambda, \Sigma^{\mu\nu}\}\Psi(x) : \rangle = \int \mathrm{d}^4k \; k^\nu \mathrm{tr}_4(\{\gamma^\lambda, \Sigma^{\mu\nu}\} W(x,k))$$

However, the momentum decomposition is independent of the particular tensors, in a word of the pseudo-gauge.

$$J_+^{\mu\nu} = \int \mathrm{d}^3\mathrm{p} \; \Big(p^\mu G_+^\nu(p) - p^\nu G^\mu(p) + \tilde{S}^{\mu\nu}(p) \Big)$$

We could have used the Belinfante pseudo-gauge

$$\widehat{T}_B^{\mu\nu} = \widehat{T}^{\mu\nu} + \widehat{T}^{\nu\mu} \qquad \widehat{\mathcal{S}}_B^{\lambda\mu\nu} = 0$$

and yet obtain the same momentum decomposition above

In the canonical pseudo-gauge, this is the only contribution to the mean spin vector

$$\tilde{S}^{\mu\nu}(p) = \int \mathrm{d}k^0 \int \mathrm{d}\Sigma_\lambda \; \mathrm{tr}_4 \left(\frac{1}{2} \{\gamma^\lambda, \Sigma^{\mu\nu}\} W_+(x,k) \right)$$

$$S^\mu(p) = -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma} p_\sigma \frac{\int \mathrm{d}k^0 \int \mathrm{d}\Sigma_\lambda \; \mathrm{tr}_4 \left(\{\gamma^\lambda, \Sigma^{\mu\nu}\} W_+(x,k) \right)}{\frac{1}{2\varepsilon} \mathrm{tr}_4 \tilde{W}(k)}$$

$$= -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma} p_\sigma \frac{\int \mathrm{d}k^0 \int \mathrm{d}\Sigma_\lambda \; \mathrm{tr}_4 \left(\{\gamma^\lambda, \Sigma^{\mu\nu}\} W_+(x,k) \right)}{\int \mathrm{d}k^0 \int \mathrm{d}\Sigma_\lambda k^\lambda \; \mathrm{tr}_4 W_+(x,k)}$$

$$S^\mu(p) = -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma} p_\sigma \frac{\int \mathrm{d}\Sigma_\lambda \; \mathrm{tr}_4 \left(\{\gamma^\lambda, \Sigma^{\mu\nu}\} W_+(x,k) \right)}{\int \mathrm{d}\Sigma_\lambda k^\lambda \; \mathrm{tr}_4 W_+(x,k)}$$

SAME RESULT as with the previous approach

The spin vector expression does not depend EXPLICITELY on a particular spin tensor, or on a particular pseudo-gauge.
However, it may implicitely depend on it through the Wigner function, i.e. on the density operator (see forthcoming Quark Matter talk)

Revisiting

F. B., V. Chandra, L. Del Zanna, E. Grossi, Ann. Phys. 338, 32 (2013),

These results deserve some discussion concerned with the physical meaning of the spin tensor, which has been the crucial ingredient to obtain (52) and the ensuing formulae. In principle, the definition of the polarization vector (50) involves the *total* angular momentum of the particle; hence the formula should be invariant under a change of the stress–energy tensor and spin tensor operators keeping the integral of the total angular momentum density invariant, a so-called *pseudo-gauge* transformation of the stress–energy tensor [22] (see also detailed discussion in Refs. [9,10]). However, in a formula like (52), one defines the local value of polarization and, therefore, a dependence on the particular spin tensor is implied. It could be expected that a space-integrated expression like (58) would be independent of the spin tensor choice; in fact this is not the case because of the explicit time-dependence of the local equilibrium density operator (see Discussion in Section 4). Such a dependence does not enable us to write the mean value of the divergence of an operator as the divergence of its mean value, thus breaking the pseudo-gauge invariance of the total angular momentum. For instance, had we used the Belinfante symmetrized stress–energy tensor, the ensuing value of polarization at local thermodynamical equilibrium (58) would vanish. To summarize, the choice of a specific spin tensor operator is necessary to calculate the polarization of particles and we have chosen the canonical spin tensor (see Eq. (16), which is the same used in Ref. [23]) to calculate the polarization of electrons. Even though it might appear disturbing that polarization at local thermodynamical equilibrium depends on the particular quantum spin tensor (whence the stress–energy tensor) of the theory, it has been recently shown that in thermodynamics this is a general feature [9,10].

Wrong

This is incorrect because it is not true in general.
Its validity is limited to the approximated expression of the Wigner Function (De Groot et al.):

$$W(x,k) \simeq \frac{1}{2} \sum_{\sigma,\tau} \int \frac{\mathrm{d}^3\mathrm{p}}{\varepsilon} \, \delta^4(k-p) u_\sigma(p) f(x,p)_{\sigma\tau} \bar{u}_\tau(p) - \delta^4(k+p) v_\sigma(p) \bar{f}(x,p)_{\tau\sigma} \bar{v}_\tau(p)$$

Quantum relativistic fluid: density operator

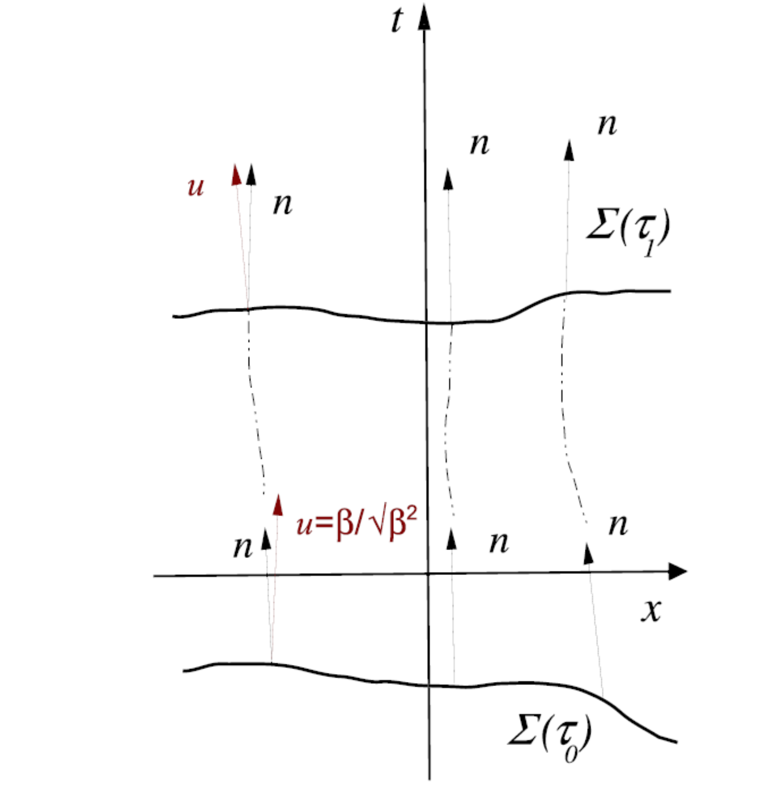
Needed to calculate the Wigner function!

$$\beta = \frac{1}{T} u$$

$$\zeta = \frac{\mu}{T}$$

General covariant
Local thermodynamic
Equilibrium density operator

$$\widehat{\rho} = \frac{1}{Z} \exp\left[- \int_\Sigma d\Sigma_\mu \left(\widehat{T}^{\mu\nu} \beta_\nu - \zeta \widehat{j}^\mu \right) \right]$$



The operator is obtained by maximizing the entropy

$$S = -\mathrm{tr}(\widehat{\rho} \log \widehat{\rho})$$

with the constraints of fixed energy-momentum density

Zubarev, 1979, Ch, Van Weert 1982

F. B., L. Bucciantini, E. Grossi, L. Tinti,
Eur. Phys. J. C 75 (2015) 191 (β frame)

T. Hayata, Y. Hidaka, T. Noumi, M. Hongo,
Phys. Rev. D 92 (2015) 065008

Global thermodynamic equilibrium

$$\widehat{\rho} = \frac{1}{Z} \exp\left[- \int_\Sigma \mathrm{d}\Sigma_\mu \, \left(\widehat{T}^{\mu\nu} \beta_\nu - \zeta \widehat{j}^\mu \right) \right]$$

Independent of the 3D hypersurface Σ if

$$\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0 \qquad\qquad \partial_\mu \zeta = 0$$

$$\beta_\mu = b_\mu + \varpi_{\mu\nu} x^\nu$$

The density operator becomes

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-b_\mu \widehat{P}^\mu + \frac{1}{2} \varpi_{\mu\nu} \widehat{J}^{\mu\nu} + \zeta \widehat{Q} \right]$$

True statistical operator (Zubarev theory)

$$\widehat{\rho} = \frac{1}{Z} \exp\left[- \int_{\Sigma(\tau_0)} \mathrm{d}\Sigma_\mu \left(\widehat{T}_B^{\mu\nu} \beta_\nu - \zeta \widehat{j}^\mu \right) \right].$$

With the Gauss theorem

$$\widehat{\rho} = \frac{1}{Z} \exp\left[- \int_{\Sigma(\tau)} \mathrm{d}\Sigma_\mu \left(\widehat{T}_B^{\mu\nu} \beta_\nu - \zeta \widehat{j}^\mu \right) + \int_{\Theta} \mathrm{d}\Theta \, \left(\widehat{T}_B^{\mu\nu} \nabla_\mu \beta_\nu - \widehat{j}^\mu \nabla_\mu \zeta \right) \right],$$

Local equilibrium, non-dissipative terms

Dissipative terms

$$W(x,k) = \mathrm{Tr}(\widehat{\rho} \widehat{W}(x,k))$$

Local thermodynamic equilibrium

$$\widehat{\rho}_{\rm LE} = \frac{1}{Z} \exp\left[- \int_\Sigma {\rm d}\Sigma_\mu \, \widehat{T}^{\mu\nu} \beta_\nu - \zeta \widehat{j}^\mu \right]$$

$$W(x,k)_{\rm LE} = (\widehat{\rho}_{\rm LE} \widehat{W}(x,k))$$

Hydrodynamic limit: Taylor expand the field β and ζ starting from *x*

$$\widehat{\rho}_{\rm LE} = \frac{1}{Z} \exp[-\beta(x)_\mu \widehat{P}^\mu + \frac{1}{2}(\partial_\mu \beta_\nu(x) - \partial_\nu \beta_\mu(x)) \widehat{J}^{\mu\nu}_x + \ldots]$$

$$\varpi_{\mu\nu} = -\frac{1}{2}(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu)$$

Thermal vorticity

Adimensional in natural units

Local thermodynamic equilibrium: leading order

$$W(x,k) = \frac{1}{Z}\mathrm{Tr}\left(\exp[-\beta(x)\cdot\widehat{P} + \frac{1}{2}\varpi(x) : \widehat{J}]\widehat{W}(x,k)\right)$$

The exact solution at global equilibrium is still missing

Educated *ansatz* of the Wigner function of the Dirac field at global equilibrium
F. B., V. Chandra, L. Del Zanna, E. Grossi, Ann. Phys. 338, 32 (2013)

$$W(x,k) \simeq \frac{1}{2}\sum_{\sigma,\tau}\int \frac{\mathrm{d}^3\mathrm{p}}{\varepsilon}\,\delta^4(k-p)u_\sigma(p)f(x,p)_{\sigma\tau}\bar{u}_\tau(p) - \delta^4(k+p)v_\sigma(p)\bar{f}(x,p)_{\tau\sigma}\bar{v}_\tau(p)$$

$$f(x,p) = \frac{1}{2m}\bar{U}(p)\left(\exp[\beta(x)\cdot p - \xi(x)]\exp[-\frac{1}{2}\varpi(x):\Sigma] + I\right)^{-1} U(p)$$

See also R. H. Fang, L. G. Pang, Q. Wang, X. N. Wang, Phys. Rev. C 94 (2016) 024904
W. Florkowski, A. Kumar and R. Ryblewski, Phys. Rev. C 98 (2018) 044906
Y. C. Liu, L. L. Gao, K. Mameda and X. G. Huang, Phys. Rev. D 99 (2019) 085014
N. Weickgenannt, X. L. Sheng, E. Speranza, Q. Wang and D. H. Rischke, Phys. Rev. D 100 (2019) 056018

$$S^\mu(p) = \frac{1}{8m}\epsilon^{\mu\nu\rho\sigma}p_\sigma\frac{\int_\Sigma \mathrm{d}\Sigma_\tau p^\tau n_F(1-n_F)\partial_\nu\beta_\rho}{\int_\Sigma \mathrm{d}\Sigma_\tau p^\tau\, n_F}$$

Conclusions

- Two different methods of deriving the mean spin vector of single particle in QFT lead to the same result
- The polarization of single particle does NOT explicitely depend on the specific set of tensors chosen to describe angular momentum density (pseudo-gauge)
- Wigner function should be evaluated at local thermodynamic equilibrium
- Exact form of the Wigner function at non-trivial global equilibrium yet unknown

Pseudo-gauge transformations

In quantum field theory there are conserved currents arising from Noether theorem (canonical currents):

$$\partial_\mu \widehat{T}^{\mu\nu} = 0$$

$$\partial_\lambda \widehat{\mathcal{J}}^{\lambda,\mu\nu} = \partial_\lambda \left(\widehat{\mathcal{S}}^{\lambda,\mu\nu} + x^\mu \widehat{T}^{\lambda\nu} - x^\nu \widehat{T}^{\lambda\mu} \right) = \partial_\lambda \widehat{\mathcal{S}}^{\lambda,\mu\nu} + \widehat{T}^{\mu\nu} - \widehat{T}^{\nu\mu} = 0$$

Spin tensor

Pseudo-gauge transformation (F. W. Hehl, Rep. Mat. Phys. 9 (1976) 55)

$$\widehat{T}'^{\mu\nu} = \widehat{T}^{\mu\nu} + \frac{1}{2}\partial_\alpha \left(\widehat{\Phi}^{\alpha,\mu\nu} - \widehat{\Phi}^{\mu,\alpha\nu} - \widehat{\Phi}^{\nu,\alpha\mu} \right)$$

$$\widehat{\mathcal{S}}'^{\lambda,\mu\nu} = \widehat{\mathcal{S}}^{\lambda,\mu\nu} - \widehat{\Phi}^{\lambda,\mu\nu}$$

Leave conservation equations and P, J unchanged

Special case: Belinfante symmetrized stress-energy tensor, spin tensor vanishing.
Tacitly understood in relativistic hydrodynamics

$$\widehat{T}_{\rm B}^{\mu\nu} = \widehat{T}^{\mu\nu} + \frac{1}{2}\partial_\alpha \left(\widehat{\mathcal{S}}^{\alpha,\mu\nu} - \widehat{\mathcal{S}}^{\mu,\alpha\nu} - \widehat{\mathcal{S}}^{\nu,\alpha\mu} \right)$$

$$\widehat{\mathcal{S}}'^{\lambda,\mu\nu} = 0$$