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Axial kinetic theory and spin transport for relativistic fermions

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Reference :
Koichi Hattori (YITP) , Yoshimasa Hidaka (RIKEN), DY,
arXiv:1903.01653

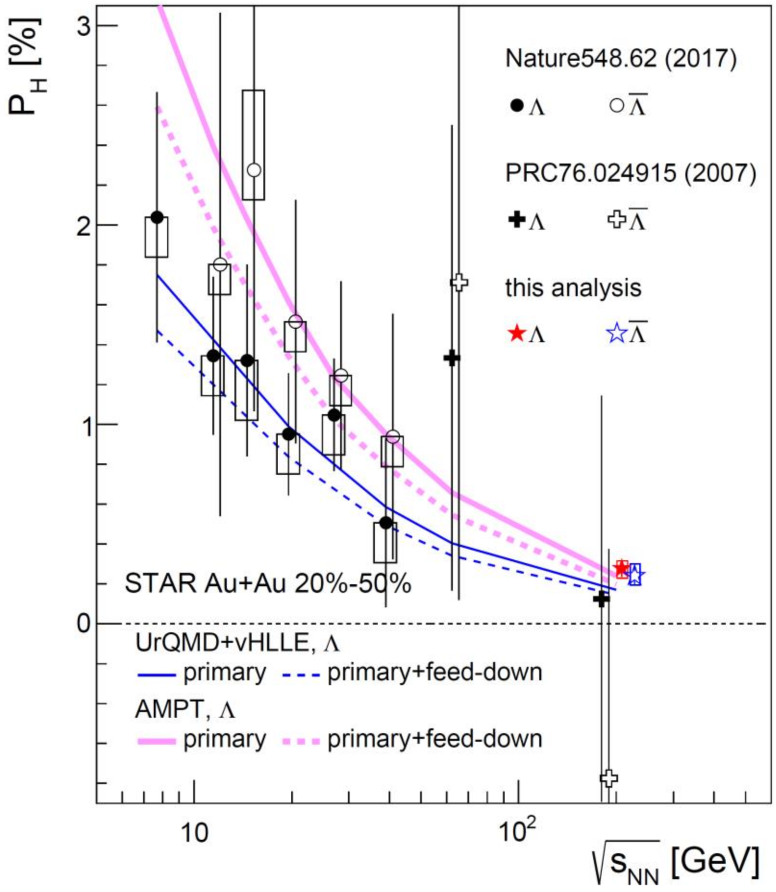
Outline

- Background and motivations in theory and phenomenology
- Axial kinetic theory (AKT) : collisionless quantum kinetic theory for tracking charge/spin transport of spin-1/2 fermions
- Comparison with related works
- Preliminary study : inclusion of collisions
- Summary & outlook

Rotating fluids with spin

- Global polarization of Λ hyperons : (see Xia's talk)

STAR, Nature 548 (2017) 62-65



STAR, PRC, 18

- Statistical model/Wigner-function approach (in equilibrium):

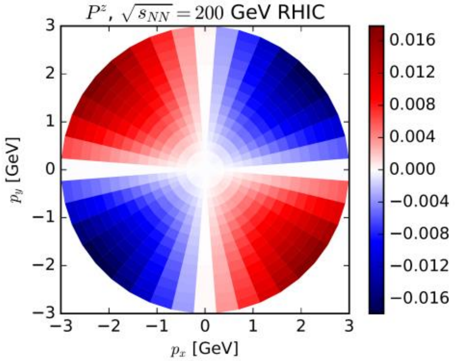
F. Becattini, et.al. 13

R. Fang, L.-G. Pang, Q. Wang, X.-N. Wang, 16

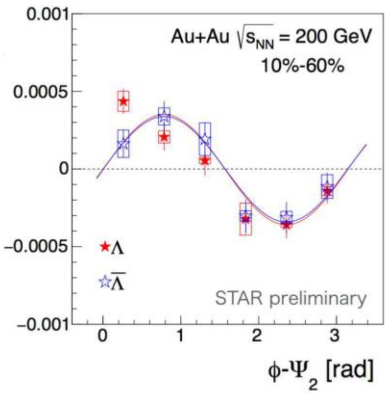
$$\mathcal{P}^\mu(q) \approx \frac{1}{8m}\epsilon^{\sigma\mu\nu\rho} q_\sigma \frac{\int d\Sigma \cdot q\omega_{\nu\rho} f_q^{(0)}(1-f_q^{(0)})}{\int d\Sigma \cdot q f_q^{(0)}},$$

$$\omega_{\nu\rho} = \frac{1}{2}\Big(\partial_\rho(u_\nu/T) - \partial_\nu(u_\rho/T)\Big).$$

- Sign problem for local polarization :



v.s.



F. Becattini, I. Karpenko, 17

(same structure, opposite signs!)

Evolution of the spin

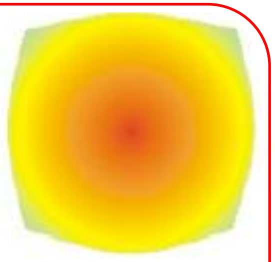
- (local) polarization may not be solely contributed by thermal vorticity

  ⇨ non-equilibrium effects may play a role

  W. Florkowski, et. al, 19
  H.-Z. Wu, L.-G. Pang, X.-G. Huang, Q. Wang, 19
- How does the spin polarization of partons (s quark) evolve?
- Current theoretical studies :

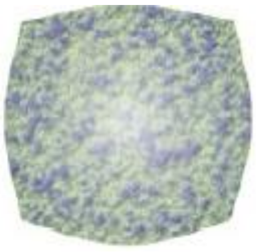
Initial states

pre-equilibrium
phase/thermaliation

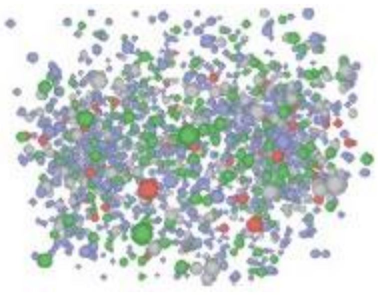


QGP

in between?



hadronization/
freeze out



hadronic gas

Initial polarization :
Hard scattering with
$b \neq 0$
Z.-T. Liang, X.-N. Wang, 05

macroscopic : spin hydro.
(Florkowski & Taya's talks)

Polarization of hadrons
in equilibrium :
e.g. statistical model
F. Becattini, et al. 13

Final polarization :
Observed in exp.

"Quantum kinetic theory (QKT) for spin transport" (microscopic theory, non-equilibrium, weak EM fields, weakly coupled)

(see also Liu & Weickgenannt's talks)

Quantum kinetic theory for fermions

- ■ QKT for massless fermions : chiral kinetic theory (CKT)
- ❑ Modified Boltzmann (Vlasov) equation with the chiral anomaly & spin-orbit int.
- ❖ Non-field theory construction : Berry phase
  - D. T. Son and N. Yamamoto, 12
  - M. Stephanov and Y. Yin, 12
  - J.-Y. Chen, et al. 14, 15
- ❖ QFT derivation : Wigner functions (WFs)
  - J.-W. Chen, S. Pu, Q. Wang, X.-N. Wang, 12
  - D. T. Son & N. Yamamoto, 12
- ➢ Covariant CKT in an arbitrary frame with BF & collisions Hidaka, Pu, DY, 16, 17

- ■ QKT for massive fermions ?
- ➢ Spin is no longer enslaved by chirality : a new dynamical dof
- ➢ To track both vector/axial charges and spin polarization
- ➢ To reproduce CKT in the massless limit
- ❖ Axial kinetic theory (AKT) : a scalar + an axial-vector equations

K. Hattori, Y. Hidaka, DY, arXiv:1903.01653 (in an arbitrary frame)

❑ similar works in the rest frame ⇨ become invalid with small mass

N. Weickgenannt, X. L. Sheng, E. Speranza, Q. Wang and D. H. Rischke, 19 (see Weickgenannt's talk)

J. H. Gao and Z. T. Liang, 19

Relativistic angular momentum

- Relativistic angular momentum for QCD (QED) : $M^{\mu\nu\lambda} = M_q^{\mu\nu\lambda} + M_g^{\mu\nu\lambda}$

(see also Becattini & Fukushima's talks)

- gauge-inv. version :

(M. Wakamatsu, 10
review : E. Leader & C. Lorce, 13)

(EOM+pseudo-gauge transf. of Ji's
decomposition (Belinfante). X. Ji, 96)

$$M_q^{\mu\nu\lambda} = \frac{1}{2}\,\epsilon^{\mu\nu\lambda\beta}\,\bar\psi\,\gamma_\beta\,\gamma_5\,\psi \;+\; \bar\psi\,\gamma^\mu\,(x^\nu\,i\,D^\lambda \;-\; x^\lambda\,i\,D^\nu\,)\,\psi,$$

$$M_g^{\mu\nu\lambda} = 2\,\mathrm{Tr}\,[\,x^\nu\,F^{\mu\alpha}\,F_\alpha{}^\lambda \;-\; x^\lambda\,F^{\mu\alpha}\,F_\alpha{}^\nu\,] \;-\; \frac{1}{2}\,\mathrm{Tr}\,F^2\,[\,x^\nu\,g^{\mu\lambda} \;-\; x^\lambda\,g^{\mu\nu}\,].$$

- Fermionic part :

$$M_q^{\lambda\mu\nu} = M_{\text{spin}}^{\lambda\mu\nu} + M_{\text{orbit}}^{\lambda\mu\nu},$$

$$M_{\text{spin}}^{\lambda\mu\nu} = \frac{\hbar}{2}\bar\psi\{\gamma^\lambda,\Sigma^{\mu\nu}\}\psi = -\frac{\hbar}{2}\epsilon^{\lambda\mu\nu\rho}\boxed{\bar\psi\gamma_5\gamma_\rho\psi,}$$

(spin polarization~
the axial-charge current (density))

see e.g. R. Fang, L.-G. Pang, Q. Wang, X.-N. Wang, 16

$$M_{\text{orbit}}^{\lambda\mu\nu} = \frac{i\hbar}{2}\bar\psi\gamma^\lambda\left(x^\mu\overleftrightarrow{D}^\nu - x^\nu\overleftrightarrow{D}^\mu\right)\psi = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu} + x^\mu T_A^{\lambda\nu} - x^\nu T_A^{\lambda\mu}$$

$$\bar T^{\mu\nu} = T^{\mu\nu} + T_A^{\mu\nu}, \quad T^{\mu\nu} = \frac{i\hbar}{4}\bar\psi\gamma^{\{\mu}\overleftrightarrow{D}^{\nu\}}\psi, \quad T_A^{\mu\nu} = \frac{i\hbar}{4}\bar\psi\gamma^{[\mu}\overleftrightarrow{D}^{\nu]}\psi$$

- EM & AM cons. : $\partial_\mu \bar T^{\mu\nu} = 0,$ ⇒ spin $\boxed{-\frac{\hbar}{2}\epsilon^{\lambda\mu\nu\rho}\partial_\lambda J_{5\rho}} + \boxed{2T_A^{\mu\nu}} = 0$ orbit

$$\partial_\lambda M_q^{\lambda\mu\nu} = 0.$$

(see also DY,18 for the analysis in CKT)

Vector/axial bases

- Wigner functions (WFs) : $\grave{S}^<(q,X) = \int d^4Y e^{\frac{iq\cdot Y}{\hbar}} \langle\bar{\psi}(X-Y/2)\psi(X+Y/2)\rangle$
- Kadanoff-Baym eq. : $(\not{\Pi} - m)\grave{S}^< + \gamma^\mu i\frac{\hbar}{2}\nabla_\mu \grave{S}^< = 0,$ $\nabla_\mu = \Delta_\mu + \mathcal{O}(\hbar^2), \quad \Pi^\mu = q^\mu + \mathcal{O}(\hbar^2)$

  (with BF) $\Delta_\mu = \partial_\mu + F_{\nu\mu}\partial/\partial q_\nu$
- Decomposition : D. Vasak, M. Gyulassy, and H. T. Elze, 87

$$\grave{S}^< = \mathcal{S} + i\mathcal{P}\gamma^5 + \mathcal{V}^\mu\gamma_\mu + \mathcal{A}^\mu\gamma^5\gamma_\mu + \frac{\mathcal{S}^{\mu\nu}}{2}\Sigma_{\mu\nu}, \quad \Sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu,\gamma_\nu]. \Longrightarrow J^\mu_{V/5} = 4\int_q (\mathcal{V}/\mathcal{A})^\mu$$

- Reducing redundant dof : replacing $\mathcal{S}$, $\mathcal{P}$, and $\mathcal{S}^{\mu\nu}$ in terms of $\mathcal{V}^\mu$ and $\mathcal{A}^\mu$.

  e.g. $m\mathcal{P} = -\frac{\hbar}{2}\nabla_\mu\mathcal{A}^\mu \Longrightarrow \partial_\mu J^\mu_5 = \frac{\hbar\mathbf{E}\cdot\mathbf{B}}{2\pi^2} + 2im\bar{\psi}\gamma_5\psi$ anomaly eq.
- Master equations (collisionless) :

$$\Delta\cdot\mathcal{V} = 0, \qquad q\cdot\mathcal{A} = 0,$$

$$(q^2 - m^2)\mathcal{V}_\mu = -\hbar\tilde{F}_{\mu\nu}\mathcal{A}^\nu, \qquad (q^2 - m^2)\mathcal{A}^\mu = \frac{\hbar}{2}\epsilon^{\mu\nu\rho\sigma}q_\sigma\Delta_\nu\mathcal{V}_\rho,$$

$$\boxed{q_\nu\mathcal{V}_\mu - q_\mu\mathcal{V}_\nu = \frac{\hbar}{2}\epsilon_{\mu\nu\rho\sigma}\Delta^\rho\mathcal{A}^\sigma}, \qquad q\cdot\Delta\mathcal{A}^\mu + F^{\nu\mu}\mathcal{A}_\nu = \frac{\hbar}{2}\epsilon^{\mu\nu\rho\sigma}(\partial_\sigma F_{\beta\nu})\partial^\beta_q\mathcal{V}_\rho$$

$\Longrightarrow -\frac{\hbar}{2}\epsilon^{\lambda\mu\nu\rho}\partial_\lambda J_{5\rho} + 2T^{\mu\nu}_A = 0$ AM conservation (spin-orbit int.)

Leading-order kinetic equations

- Perturbative solution : $(\mathcal{V}/\mathcal{A})^\mu \;=\; (\mathcal{V}/\mathcal{A})^\mu_0 + \hbar(\mathcal{V}/\mathcal{A})^\mu_1$

- Leading order (LO) : $(\mathcal{V}_0/\mathcal{A}_0)^\mu = 2\pi(q/a)^\mu\delta(q^2-m^2)f_{V/A}$
  - Dynamical variables : $f_{V/A}(q,X)$ & $a^\mu(q,X)$
  - Spin four vector $a^\mu(q,X)$: $q\cdot a = q^2 - m^2$ (vanishes on-shell)

$$m = 0 \Longrightarrow a^\mu \;=\; q^\mu$$ (spin enslavement)

- LO kinetic theory :

Vlasov Eq. : $$0 = \delta(q^2-m^2)q\cdot\Delta f_V, \quad \Delta_\mu \;=\; \partial_\mu + F_{\nu\mu}\partial/\partial q_\nu$$

BMT Eq. : $$0 = \delta(q^2-m^2)\Big(q\cdot\Delta(a^\mu f_A) + F^{\nu\mu}a_\nu f_A\Big)$$

Bargmann-Michel-Telegdi, 59

(off-shell, g = 2)

$m = 0$: BMT Eq. $\Longrightarrow$ $$0 = \delta(q^2)q^\mu q\cdot\Delta f_A$$

Collisionless WFs for massive fermions

- WFs up to $\mathcal{O}(\hbar^1)$:

$$\mathcal{V}^\mu = 2\pi\Big[\delta(q^2-m^2)\big(q^\mu f_V + \hbar G^\mu\big) + \hbar\tilde{F}^{\mu\nu}a_\nu\delta'(q^2-m^2)f_A\Big],$$

$$\mathcal{A}^\mu = 2\pi\Big[\delta(q^2-m^2)\big(a^\mu f_A + \hbar H^\mu\big) + \hbar\tilde{F}^{\mu\nu}q_\nu\delta'(q^2-m^2)f_V\Big],$$

Magnetization currents (spin-orbit int.) :

$$G^\mu = \frac{\epsilon^{\mu\nu\rho\sigma}n_\nu}{2q\cdot n}\left[\Delta_\rho(a_\sigma f_A) + F_{\rho\sigma}f_A\right].$$

$$\boxed{H^\mu = \frac{\epsilon^{\mu\nu\alpha\beta}q_\alpha n_\beta}{2a\cdot n}\Delta_\nu f_V.}$$

$$\xrightarrow{\quad\quad}$$
$$m = 0$$

Side-jump terms : for CVE

Chen et al. 14.
Hidaka, Pu, DY, 16

$$(G/H)^\mu = \frac{\epsilon^{\mu\nu\alpha\beta}q_\alpha n_\beta}{2q\cdot n}\Delta_\nu f_{V/A}$$

$$f_{V/A} = f_R \pm f_L$$

obtained from the wave functions for
free Dirac spinors instead of KB equations

- Modified frame transformation :

$$f_V^{(n')} - f_V^{(n)} = \frac{\hbar\epsilon^{\lambda\nu\rho\sigma}n_\lambda n'_\nu}{2(q\cdot n)(q\cdot n')}(\Delta_\rho a_\sigma + F_{\rho\sigma})f_A,$$

$$a^{(n')\mu}f_A^{(n')} - a^{(n)\mu}f_A^{(n)} = \frac{\hbar\epsilon^{\mu\nu\alpha\beta}}{2}\left(\frac{n_\beta}{(q\cdot n+m)} - \frac{n'_\beta}{(q\cdot n'+m)}\right)q_\alpha\Delta_\nu f_V$$

- The rest frame : $n^\mu = q^\mu/m$ $\xrightarrow[m=0]{\quad\quad}$ divergence of G^μ

N. Weickgenannt, et al, 19
J. H. Gao and Z. T. Liang, 19

Axial kinetic theory

- AKT in an arbitrary spacetime-dep. frame $(n^\mu = n^\mu(X))$:
- Scalar kinetic equation (SKE): remaining in the massless limit $\xrightarrow{m=0}$ CKT

$$0 = \boxed{\delta(q^2-m^2)\Big[q\cdot\Delta f_V + \hbar\Big(\frac{E_\mu S^{\mu\nu}_{a(n)}}{q\cdot n}\Delta_\nu + S^{\mu\nu}_{a(n)}(\partial_\mu F_{\rho\nu})\partial^\rho_q + \big(\partial_\mu S^{\mu\nu}_{a(n)}\big)\Delta_\nu\Big)f_A\Big] - \frac{\delta'(q^2-m^2)}{q\cdot n}B^\mu\Box_{\mu\nu}\tilde{a}^\nu}$$

$$+\frac{\hbar}{2}\delta(q^2-m^2)\epsilon^{\mu\nu\alpha\beta}\left[\Delta_\mu\left(\frac{n_\beta}{q\cdot n}\right)[(\Delta_\nu a_\alpha)+F_{\nu\alpha}] + \frac{n_\beta}{q\cdot n}\Big((\partial_\mu F_{\rho\nu})(\partial^\rho_q a_\alpha) + \Big[(\Delta_\nu a_\alpha) - F_{\rho\nu}\big(\partial^\rho_q a_\alpha\big)\Big]\Delta_\mu\Big)\right]f_A,$$

$$\tilde{a}^\mu = a^\mu f_A,\quad \Box_{\mu\nu}\tilde{a}^\nu = q\cdot\Delta\tilde{a}_\mu + F_{\nu\mu}\tilde{a}^\nu,\quad S^{\mu\nu}_{a(n)} = \frac{\epsilon^{\mu\nu\alpha\beta}a_\alpha n_\beta}{2q\cdot n},\quad S^{\mu\nu}_{m(n)} = \frac{\epsilon^{\mu\nu\alpha\beta}q_\alpha n_\beta}{2(q\cdot n+m)} = \frac{\epsilon^{\mu\nu\alpha\beta}q_\alpha n_\beta}{2a\cdot n}.$$

- Axial-vector kinetic equation (AKE) :

remaining in the massless limit

BMT Eq

$$0 = \boxed{\delta(q^2-m^2)\Big(q\cdot\Delta(a^\mu f_A) + F^{\nu\mu}a_\nu f_A\Big)} + \boxed{\hbar q^\mu\Big\{\delta(q^2-m^2)\Big[(\partial_\alpha S^{\alpha\nu}_{m(n)})\Delta_\nu + \frac{S^{\alpha\nu}_{m(n)}E_\alpha\Delta_\nu}{q\cdot n+m} + S^{\rho\nu}_{m(n)}(\partial_\rho F_{\beta\nu})\partial^\beta_q\Big]}$$

$$\boxed{-\delta'(q^2-m^2)\frac{q\cdot B}{q\cdot n+m}q\cdot\Delta\Big\}f_V} + \hbar m\Big\{\frac{\delta(q^2-m^2)\epsilon^{\mu\nu\alpha\beta}}{2(q\cdot n+m)}\Big[m(\partial_\alpha n_\beta)\Delta_\nu + (mn_\beta+q_\beta)\Big(\frac{(E_\alpha-\partial_\alpha(q\cdot n))}{q\cdot n+m}\Delta_\nu$$

$$-(\partial_\nu F_{\rho\alpha})\partial^\rho_q\Big)\Big] + \delta'(q^2-m^2)\frac{(mn_\beta+q_\beta)\tilde{F}^{\mu\beta}}{q\cdot n+m}q\cdot\Delta\Big\}f_V.$$

$\xrightarrow{m=0}$ q^μ CKT

spin enslavement by chirality & momentum

AKT with collisions?

- To include collisions in AKT (preliminary) (with Hattori & Hidaka)
- KB eq. with collisions : $(\not{\Pi} - m)S^< + \gamma^\mu i\frac{\hbar}{2}\nabla_\mu S^< = \frac{i\hbar}{2}\left(\Sigma^< \star S^> - \Sigma^> \star S^<\right)$

⇩

AKT : $\square^{(n)}\mathcal{A}^\mu = \hat{\mathcal{C}}^\mu_{\rm cl} + \hbar\hat{\mathcal{C}}^{(n)\mu}_{\rm Q}$

(spin diffusion) (spin polarization)

- "Classical" ($\mathcal{O}(\hbar^0)$) spin diffusion in weakly-coupled QGP (leading log)
- In our framework up to $\mathcal{O}(\hbar^0)$: S. Li, H.-U. Yee, 19

SKE : $$0 = \delta(q^2-m^2)\Big\{q\cdot\partial f_{Vq} - \frac{g_c^2C_2(F)m_D^2}{8\pi}\ln(1/g_c)\Big[2(1-f_{Vq}) + \frac{E_q^2}{|\mathbf{q}|^2}\left(1-\frac{m^2\eta_q}{q_0|\mathbf{q}|}\right)(1-2f_{Vq})q_\perp^\beta\partial_{q_\perp^\beta}$$
$$+\frac{m^2T}{2|\mathbf{q}|^3}\left(\left(1-\frac{3E_q^2}{|\mathbf{q}|^2}\right)\eta_q + \frac{3E_q}{|\mathbf{q}|}\right)q_\perp^\alpha q_\perp^\beta\partial_{q_\perp^\alpha}\partial_{q_\perp^\beta} - \frac{E_qT}{2}\left(\left(3-\frac{E_q^2}{|\mathbf{q}|^2}\right) + \frac{m^4\eta_q}{E_q|\mathbf{q}|^3}\right)\eta^{\alpha\beta}\partial_{q_\perp^\alpha}\partial_{q_\perp^\beta}\Big]f_{Vq}\Big\}$$

(agrees with Li & Yee except for nonlinear terms in f_{Vq} : f_{Vq} → FD distribution in equilibrium)

AKE : $$0 = \delta(q^2-m^2)\Big[q\cdot\partial\tilde{a}_q^\mu + \frac{g_c^2C_2(F)m_D^2}{8\pi E_q}\ln(1/g_c)\Big(\tilde{a}_q^\mu\grave{\mathcal{Q}}_{cl}^{(1)} + u^\mu\grave{\mathcal{Q}}_{cl}^{(2)} + \hat{q}_\perp^\mu\grave{\mathcal{Q}}_{cl}^{(3)} + \grave{\mathcal{Q}}_{cl}^{(4)}\hat{q}_\perp^\nu\partial_{q_{\perp\mu}}\tilde{a}_{q\nu} + \grave{\mathcal{Q}}_{cl}^{(5)}\hat{q}^\nu\partial_{q_\perp^\nu}\tilde{a}_q^\mu$$
$$+\grave{\mathcal{Q}}_{cl}^{(6)}\eta^{\nu\rho}\partial_{q_\perp^\nu}\partial_{q_\perp^\rho}\tilde{a}_q^\mu + \grave{\mathcal{Q}}_{cl}^{(7)}\hat{q}_\perp^\nu\hat{q}_\perp^\rho\partial_{q_\perp^\nu}\partial_{q_\perp^\rho}\tilde{a}_q^\mu\Big)\Big]$$

$\tilde{a}_q^\mu \equiv a_q^\mu f_{Aq}$

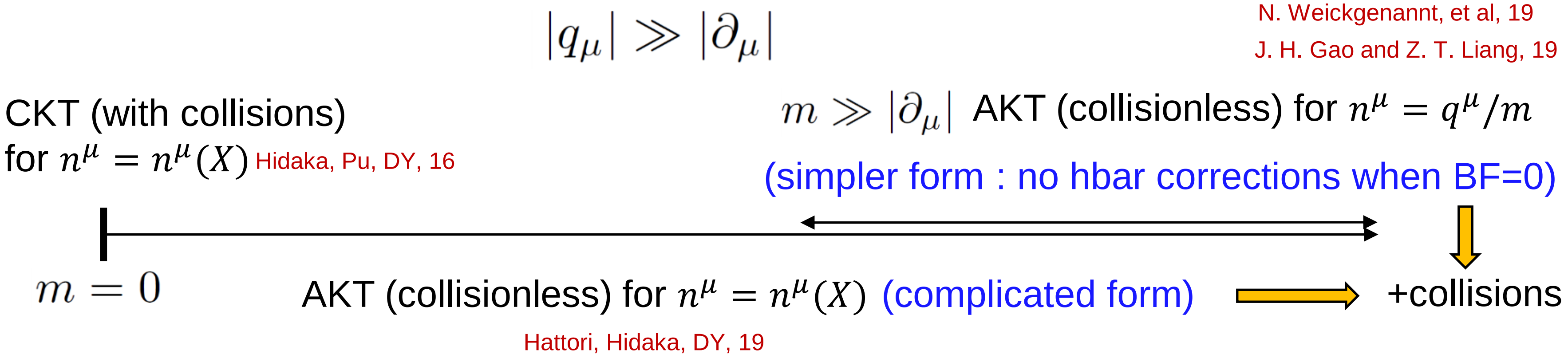
$\xrightarrow{m\,=\,0}$ consistent with the SKE

(spin parameterization differs from Li & Yee : covariant form)

Summary & outlook

- AKT provides a theoretical framework to track the entangled dynamics of charges and spin for spin-1/2 fermions with arbitrary mass.

- Current status of the QKT for relativistic fermions under BF :



$$|q_\mu| \gg |\partial_\mu|$$

N. Weickgenannt, et al, 19

J. H. Gao and Z. T. Liang, 19

CKT (with collisions)
for $n^\mu = n^\mu(X)$ Hidaka, Pu, DY, 16

$m \gg |\partial_\mu|$ AKT (collisionless) for $n^\mu = q^\mu/m$

(simpler form : no hbar corrections when BF=0)

$m = 0$

AKT (collisionless) for $n^\mu = n^\mu(X)$ (complicated form) $\Longrightarrow$ +collisions

Hattori, Hidaka, DY, 19

- AKT with collisions : hbar terms for spin polarization

- Even the spin diffusion term in collisions is complicated. How to simplify it and make future simulations practical ?

Thank you!

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Further comments on AKT

- WFs are “frame independent” though the wave-function parts and distribution functions therein are both frame dependent.
- Solving AKT for $f_{V/A}(q,X)$ & $a^\mu(q,X)$ with a proper choice of n^μ.
- Using the WFs to compute the field-theory defined observables :

vector/axial-charge currents :

(anti-)symmetric energy-momentum tensors :

$$J^\mu_{V/5} = 4\int_q (\mathcal{V}/\mathcal{A})^\mu, \qquad T^{\mu\nu}_{S/A} = 2\int_q \left(\mathcal{V}^\mu q^\nu \pm \mathcal{V}^\nu q^\mu\right), \qquad \int_q \equiv \int d^4q/(2\pi)^4.$$

- The anti-symmetric EM tensor is responsible for angular-momentum transfer (via spin-orbit coupling) :

$$\partial_\lambda M_C^{\lambda\mu\nu} = 0. \quad \Rightarrow \quad \overset{\text{spin}}{\boxed{-\frac{\hbar}{2}\epsilon^{\lambda\mu\nu\rho}\partial_\lambda J_{5\rho}}} + \overset{\text{orbit}}{\boxed{2T_A^{\mu\nu}}} = 0$$

(AM conservation)

(see also DY,18 for the analysis with $m = 0$)

already captured by one of master Eqs., $q_\nu\mathcal{V}_\mu - q_\mu\mathcal{V}_\nu = \frac{\hbar}{2}\epsilon_{\mu\nu\rho\sigma}\Delta^\rho\mathcal{A}^\sigma$

AM conservation in global equilibrium

- Global equilibrium (no collisions $\omega^\mu \neq 0,\ B^\mu \neq 0.$) : $\partial_\mu J^\mu_{V/5} = \partial_\mu T^{\mu\nu} = 0,$
- Conservation of canonical EM & AM tensors :

$$\partial_\mu \bar{T}^{\mu\nu} = 0, \qquad \partial_\lambda M_C^{\lambda\mu\nu} = 0. \quad \Rightarrow$$

spin

orbit

$$\boxed{-\frac{\hbar}{2}\epsilon^{\lambda\mu\nu\rho}\partial_\lambda J_{5\rho}} + \boxed{2T_A^{\mu\nu}} = 0$$

- Weyl fermions :

DY, 18

$$\boxed{T_A^{\mu\nu} = \frac{\hbar}{2}N_A\left(\omega^\mu u^\nu - \omega^\nu u^\mu\right)}$$

from side-jumps

CSE & CVE

$$M^{\lambda\mu\nu}_{\mathrm{spin}}(X) = \frac{\hbar}{2}\epsilon^{\lambda\mu\nu\rho}\left(N_A u_\rho + \boxed{\hbar\sigma_{BA}B_\rho + \hbar\sigma_{\omega A}\omega_\rho\right)}$$

- ➢ $\mathcal{O}(\hbar)$: spin-orbit cancellation
- ➢ Higher orders : we need higher-order WFs.
- Near local equilibrium :

local torque even without EM fields

$$\partial_\lambda M_C^{\lambda\mu\nu} = X^{[\mu}F^{\nu]\rho}J_{V\rho} - \frac{u_\rho}{\tau_R}X^{[\mu}\delta T^{\rho\nu]} - \boxed{\frac{\hbar}{4}\partial_\lambda\left(X^{[\mu}\epsilon^{\nu]\lambda\alpha\beta}\frac{u_\alpha \delta J_{5\perp\beta}}{\tau_R}\right)}$$

WFs from free Dirac fields

- Construction from wave functions : M. Peskin and D. Schroeder, An Introduction to QFT (95)

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s u^s(p) e^{-ip\cdot x} a^s_{\mathbf{p}}, \quad u^s(p) = (\sqrt{p\cdot\sigma}\xi^s, \sqrt{p\cdot\bar{\sigma}}\xi^s)^T$$

- Lesser propagator :

$$S^<(x,y) = \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{\sqrt{2E_{\mathbf{p}'}}}$$

$$\times \sum_{s,s'} u^s(p) \bar{u}^{s'}(p') \langle a^{s'\dagger}_{\mathbf{p}'} a^s_{\mathbf{p}} \rangle e^{ip_-\cdot X - ip_+\cdot Y}, \quad \begin{aligned} p^\mu_+ &= (p+p')^\mu/2 \\ p^\mu_- &= (p-p')^\mu. \end{aligned}$$

(performing p_- expansion for u ⇨ $\hbar$ expansion)

- Parameterizing the density operators :

$$\langle a^{s'\dagger}_{\mathbf{p}'} a^s_{\mathbf{p}} \rangle = \delta_{ss'} N_V(\mathbf{p},\mathbf{p}') + \mathcal{A}_{ss'}(\mathbf{p},\mathbf{p}'), \quad \mathcal{A}_{ss'}(X,q) \neq 0 \text{ when } s \neq s'$$

spin sum : $\sum_s \xi_s \xi_s^\dagger = n\cdot\sigma = I$, $\sum_{s,s'} \xi_s \mathcal{A}_{ss'} \xi^\dagger_{s'} = S\cdot\sigma$ ⇨ $S\cdot n = 0$

- WT :

$$\tilde{f}_V(q,X) \equiv \int \frac{d^3p_-}{(2\pi)^3} N_V\left(\boldsymbol{q} + \frac{\boldsymbol{p}_-}{2}, \boldsymbol{q} - \frac{\boldsymbol{p}_-}{2}\right) e^{-ip_-\cdot X},$$

$$\hat{S}_\mu(q,X) \equiv \int \frac{d^3p_-}{(2\pi)^3} S_\mu\left(\boldsymbol{q} + \frac{\boldsymbol{p}_-}{2}, \boldsymbol{q} - \frac{\boldsymbol{p}_-}{2}\right) e^{-ip_-\cdot X}$$

Magnetization currents

- Re-parameterization :

$$f_V = \tilde{f}_V - \frac{\hbar S^{\mu\nu}_{m(n)}}{q\cdot n}\partial_\nu \hat{S}_\mu, \qquad S^{\mu\nu}_{m(n)} = \frac{\epsilon^{\mu\nu\alpha\beta}q_\alpha n_\beta}{2(q\cdot n+m)} = \frac{\epsilon^{\mu\nu\alpha\beta}q_\alpha n_\beta}{2a\cdot n}.$$

$$a\cdot n f_A = \hat{S}\cdot q, \quad a_{\perp\mu}f_A = \frac{(\hat{S}\cdot q)q_{\perp\mu}}{q\cdot n+m} - m\hat{S}_\mu \qquad v^\mu_\perp \equiv v^\mu - (v\cdot n)n^\mu$$

- Free WFs up to $\mathcal{O}(\hbar^1)$:

$$\mathcal{V}^\mu = 2\pi\delta(q^2-m^2)\Big[q^\mu f_V + \hbar\frac{\epsilon^{\mu\nu\alpha\beta}n_\beta}{2q\cdot n}\partial_\nu(a_\alpha f_A)\Big],$$

$$\mathcal{A}^\mu = 2\pi\delta(q^2-m^2)\Big[a^\mu f_A + \hbar S^{\mu\nu}_{m(n)}\partial_\nu f_V\Big]. \quad \Longrightarrow \quad H^\mu = S^{\mu\nu}_{m(n)}\Delta_\nu f_V$$

generalization

- Freedom for redefining a^μ : $\bar{a}^\mu f_A \equiv a^\mu f_A + \hbar H^\mu$

non-uniqueness of magnetization-current terms