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# Kinetic theory for massive spin-1/2 particles from the Wigner-function formalism

N. Weickgenannt, X.-l. Sheng, E. Speranza, Q. Wang, and D.H. Rischke,
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New development of hydrodynamics and its applications in Heavy-Ion Collisions
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# Towards MHD with spin

- For massive spin-0 particles, second-order dissipative MHD has already been studied.

  G.S. Denicol, X.-G. Huang, E. Molnar, G.M. Monteiro, H. Niemi, J. Noronha, D.H. Rischke, and Q. Wang, PRD 98 (2018) 076009

  G.S. Denicol, E. Molnar, H. Niemi, and D.H. Rischke, PRD 99 (2019) 056017

- What we want: kinetic theory and fluid dynamics for massive spin-1/2 particles in inhomogeneous electromagnetic fields.

  J.-H. Gao, and Z.-T. Liang, PRD 100 (2019) 056021

  K. Hattori, Y. Hidaka, and D.-L. Yang, arXiv:1903.01653 (2019)

  Z. Wang, X. Guo, S. Shi, and P. Zhuang, PRD 100 (2019) 014015

- Starting point: quantum field theory, Dirac equation.
- Strategy: use Wigner functions to derive kinetic theory.
- Goal: determine fluid-dynamical equations of motion with spin from resulting kinetic equation.

## Wigner functions

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- Quantum analogue of classical distribution function.
- Contains information about quantum state of system.
- Off-equilibrium: two-point function depends not only on relative coordinate $y$, but also on central coordinate $x$.
- Wigner transformation of two-point function:
  H.-Th. Elze, M. Gyulassy, and D. Vasak, Ann. Phys. 173 (1987) 462

$$W(x,p) = \int \frac{d^4 y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar} p \cdot y} \langle : \bar{\Psi}(x + \frac{y}{2}) \Psi(x - \frac{y}{2}) : \rangle \,,$$

# Wigner functions

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$$W(x,p) = \int \frac{d^4 y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar} p\cdot y} \langle : \bar{\Psi}(x+\frac{y}{2}) U(x+\frac{y}{2},x) U(x,x-\frac{y}{2}) \Psi(x-\frac{y}{2}) : \rangle \,,$$

with gauge link

$$U(b,a) \equiv P \exp\left( -\frac{i}{\hbar} \int_a^b dz^\mu \, A_\mu(z) \right)$$

to ensure gauge invariance.

## Transport equation

- From Dirac equation: transport equation for Wigner function:

  H.-Th. Elze, M. Gyulassy, and D. Vasak, Ann. Phys. 173 (1987) 462

$$(\gamma_\mu K^\mu - m) W(X,p) = 0$$

with

$$\begin{aligned} K^\mu &\equiv \Pi^\mu + \frac{1}{2} i\hbar \nabla^\mu , \\ \nabla^\mu &\equiv \partial_x^\mu - j_0(\Delta) F^{\mu\nu} \partial_{p\nu} , \\ \Pi^\mu &\equiv p^\mu - \hbar \frac{1}{2} j_1(\Delta) F^{\mu\nu} \partial_{p\nu} , \end{aligned}$$

$\Delta = \frac{1}{2}\hbar\partial_p \cdot \partial_x$ with $\partial_x$ only acting on $F^{\mu\nu}$ and $j_0(r) = \sin(r)/r$, $j_1(r) = [\sin(r) - r\cos(r)]/r^2$ spherical Bessel functions.

- Exact quantum kinetic equation for Wigner function for massive spin 1/2-particles and inhomogeneous fields!
- Only assumption: vanishing collision kernel, external classical gauge fields.

# Calculating the Wigner function

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- In general: result of calculation of Wigner function directly from definition is not on-shell.
- Momentum variable of directly calculated Wigner function is physical (kinetic) momentum
  for vanishing gradients, i.e., in global equilibrium without rotation,
  or in the classical limit ($\hbar = 0$).
- But we want:
  inhomogeneous phase-space distribution,
  quantum effects.
- Idea: Find general solutions of transport equation for Wigner function by expanding in powers of $\hbar$.
- For zeroth order use results of direct calculation.

# Strategy

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- Decompose $W$ in transport equation into generators of Clifford algebra:

$$W = \frac{1}{4}\left(\mathcal{F} + i\gamma^5\mathcal{P} + \gamma^\mu\mathcal{V}_\mu + \gamma^5\gamma^\mu\mathcal{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}\right).$$

- Insert into transport equation for Wigner function.
- Get system of 32 coupled (differential) equations.
- Equations for $\mathcal{F}$ (scalar, “distribution function”) and $\mathcal{S}_{\mu\nu}$ (tensor, “dipole moment”) decouple from rest.
- Solve by expanding in powers of $\hbar$, assuming that Wigner function gradients, em field strengths and em field gradients are sufficiently small.
- Determine $\mathcal{V}_\mu$ (“vector current“), $\mathcal{A}_\mu$ ("polarization“), $\mathcal{P}$ from $\mathcal{S}_{\mu\nu}, \mathcal{F}$.
- Results will hold up to order $\mathcal{O}(\hbar)$.

# Conventions

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- Notation: $W = W^{(0)} + \hbar W^{(1)} + \mathcal{O}(\hbar^2)$.
- To simplify notation: only write positive-energy parts of solutions.
- Polarization direction $n^\mu(x,p)$: space-like unit vector parallel to axial-vector current.
- Spin quantization direction: unit vector, purely spatial in particle rest frame.
  Here: chosen to be identical to polarization direction.

$$\bar{u}_s(x,p)\gamma^\mu\gamma^5 u_s(x,p) = 2ms\, n^\mu(x,p)$$

  $\rightarrow$ Distribution function diagonal in spin indices!

$$f_{rs} = f_s \delta_{rs}$$

- Dirac spinors space-time dependent.
- Spin quantization direction in rest-frame $\mathbf{n}^\star$ space-time and momentum dependent.

$$\bar{u}_s^\star(x,p)\boldsymbol{\gamma}\gamma^5 u_s^\star(x,p) = 2ms\,\mathbf{n}^\star(x,p)$$

## Zeroth-order Wigner function

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- Direct calculation yields

$$
\begin{aligned}
\mathcal{F}^{(0)}(x,p) &= m\,\delta(p^2-m^2)V^{(0)}(x,p), \\
\mathcal{A}^{(0)}_\mu(x,p) &= m\,n^{(0)}_\mu\,\delta(p^2-m^2)A^{(0)}(x,p), \\
\mathcal{P}^{(0)}(x,p) &= 0, \\
\mathcal{V}^{(0)}_\mu(x,p) &= p_\mu\,\delta(p^2-m^2)V^{(0)}(x,p), \\
\mathcal{S}^{(0)}_{\mu\nu}(x,p) &= m\,\Sigma^{(0)}_{\mu\nu}\,\delta(p^2-m^2)A^{(0)}(x,p),
\end{aligned}
$$

with

$$
\begin{aligned}
V^{(0)}(x,p) &\equiv \frac{2}{(2\pi\hbar)^3}\sum_s f^{(0)}_s(x,p), \\
A^{(0)}(x,p) &\equiv \frac{2}{(2\pi\hbar)^3}\sum_s s f^{(0)}_s(x,p), \\
\Sigma^{(0)}_{\mu\nu} &\equiv -\frac{1}{m}\epsilon_{\mu\nu\alpha\beta}p^\alpha n^{(0)\beta}.
\end{aligned}
$$

- Solution fulfills zeroth-order transport equation for Wigner function.

# Next-to-leading order from transport equation

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- Insert zeroth-order solution into first-order transport equation.
- Determine general form of $\mathcal{F}$ and $\mathcal{S}^{\mu\nu}$ up to order $\hbar$ from constraints.
- Generalized on-shell conditions:

$$\begin{aligned}(p^2 - m^2)\mathcal{F} &= \frac{1}{2}\hbar F^{\mu\nu}\mathcal{S}_{\mu\nu} + \mathcal{O}(\hbar^2),\\ (p^2 - m^2)\mathcal{S}_{\mu\nu} &= \hbar F_{\mu\nu}\mathcal{F} + \mathcal{O}(\hbar^2).\end{aligned}$$

- with additional constraint:

$$p_\mu \mathcal{S}^{\mu\nu} = -\frac{\hbar}{2}\nabla^\nu \mathcal{F} + \mathcal{O}(\hbar^2).$$

- $\mathcal{V}^\mu$, $\mathcal{A}^\mu$ and $\mathcal{P}$ only couple to $\mathcal{F}$ and $\mathcal{S}^{\mu\nu}$:

$$\begin{aligned}\mathcal{V}^\mu &= \frac{1}{m}(p^\mu \mathcal{F} - \frac{1}{2}\hbar\nabla_\nu \mathcal{S}^{\nu\mu}) + \mathcal{O}(\hbar^2),\\ \mathcal{A}^\mu &= -\frac{1}{2m}\epsilon^{\mu\nu\alpha\beta}p_\nu \mathcal{S}_{\alpha\beta} + \mathcal{O}(\hbar^2),\\ \mathcal{P} &= -\frac{1}{2m}\hbar\nabla_\mu \mathcal{A}^\mu + \mathcal{O}(\hbar^2).\end{aligned}$$

## General results up to $\mathcal{O}(\hbar)$

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$$
\begin{aligned}
\mathcal{F} &= m\left[V\,\delta(p^2-m^2) - \frac{\hbar}{2}F^{\mu\nu}\Sigma^{(0)}_{\mu\nu}A^{(0)}\,\delta'(p^2-m^2)\right] + \mathcal{O}(\hbar^2)\,,\\
\mathcal{S}_{\mu\nu} &= m\left[\bar{\Sigma}_{\mu\nu}\delta(p^2-m^2) - \hbar F_{\mu\nu}V^{(0)}\delta'(p^2-m^2)\right] + \mathcal{O}(\hbar^2)\,,\\
\mathcal{P} &= \frac{\hbar}{4m}\epsilon^{\mu\nu\alpha\beta}\nabla_\mu\left[p_\nu\Sigma^{(0)}_{\alpha\beta}A^{(0)}\,\delta(p^2-m^2)\right] + \mathcal{O}(\hbar^2)\,,\\
\mathcal{V}_\mu &= \delta(p^2-m^2)\left[p_\mu V + \frac{\hbar}{2}\nabla^\nu\Sigma^{(0)}_{\mu\nu}A^{(0)}\right]\\
&\quad -\hbar\left[\frac{1}{2}p_\mu F^{\alpha\beta}\Sigma^{(0)}_{\alpha\beta} + \Sigma^{(0)}_{\mu\nu}F^{\nu\alpha}p_\alpha\right]A^{(0)}\,\delta'(p^2-m^2) + \mathcal{O}(\hbar^2)\,,\\
\mathcal{A}_\mu &= -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}p^\nu\bar{\Sigma}^{\alpha\beta}\,\delta(p^2-m^2) + \hbar\tilde{F}_{\mu\nu}p^\nu V^{(0)}\,\delta'(p^2-m^2) + \mathcal{O}(\hbar^2)\,,
\end{aligned}
$$

with

$$
\begin{aligned}
\bar{\Sigma}^{(0)\mu\nu} &= \Sigma^{(0)\mu\nu}A^{(0)},\\
p_\nu\bar{\Sigma}^{\mu\nu} &= \hbar\nabla^\mu V^{(0)}.
\end{aligned}
$$

# So what does this mean?

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- $V$ and $\bar{\Sigma}^{\mu\nu}$ have to be determined through kinetic equations and constraint equation.
- By Taylor expansion of $\delta$-function:

$$\mathcal{F} = \frac{2}{(2\pi\hbar)^3} m \sum_s \delta\left(p^2 - m^2 - \hbar \frac{s}{2} \Sigma^{(0)}_{\mu\nu} F^{\mu\nu}\right) f_s .$$

  Modified on-shell condition.
  $f_s$ distribution functions for spin-up ($s=+$) and spin-down ($s=-$) particles, $V = f_+ + f_-$ and $A^{(0)} = f_+^{(0)} - f_-^{(0)}$.
- Write

$$\bar{\Sigma}^{\mu\nu} \equiv \Xi^{\mu\nu} + \frac{\hbar}{2} \chi^{\mu\nu}$$

  with

$$p_\nu \Xi^{\mu\nu} = 0,$$

  "classical" dipole moment;

$$p_\nu \chi^{\mu\nu} = \nabla^\mu V^{(0)},$$

  dipole moment induced by gradients.

# Kinetic equations

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- Kinetic equations for $V$ and $\bar{\Sigma}_{\mu\nu}$:

$$
\begin{aligned}
0 &= \delta(p^2 - m^2)\left[p\cdot\nabla V + \frac{\hbar}{4}(\partial_x^\alpha F^{\mu\nu})\partial_{p\alpha}\bar{\Sigma}_{\mu\nu}\right] \\
&\quad -\frac{\hbar}{2}\delta'(p^2 - m^2)F^{\alpha\beta}\, p\cdot\nabla\bar{\Sigma}_{\alpha\beta} + \mathcal{O}(\hbar^2), \\
0 &= \delta(p^2 - m^2)\left[p\cdot\nabla\bar{\Sigma}_{\mu\nu} - F^{\alpha}_{\ [\mu}\bar{\Sigma}_{\nu]\alpha} + \frac{\hbar}{2}(\partial_{x\alpha}F_{\mu\nu})\partial_p^\alpha V\right] \\
&\quad -\hbar\delta'(p^2 - m^2)F_{\mu\nu}\, p\cdot\nabla V + \mathcal{O}(\hbar^2).
\end{aligned}
$$

- Can we "get rid of" $\delta'$-terms?

$$
\hbar\,\delta(p^2 - m^2)p\cdot\nabla V \quad \in \quad \mathcal{O}(\hbar^2).
$$

# Omitting off-shell term

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- Wigner function and kinetic equations invariant under transformation:

$$\begin{aligned} V &\rightarrow \hat{V} = V + (p^2 - m^2)\delta V, \\ \bar{\Sigma}_{\mu\nu} &\rightarrow \hat{\bar{\Sigma}}_{\mu\nu} = \bar{\Sigma}_{\mu\nu} - \hbar F_{\mu\nu}\delta V. \end{aligned}$$

- Find transformation such that

$$\int dp^0\, \delta'(p^2 - m^2) G(x,p)\, p \cdot \nabla \hat{V} \in \mathcal{O}(\hbar)$$

  for arbitrary $G(x,p)$.
- Analogously for $p \cdot \nabla \bar{\Sigma}_{\alpha\beta}$.
- Drop "$\delta'$-terms" in kinetic equations without loss of generality!

# Boltzmann equation for massive spin-1/2 particles

- Generalized Boltzmann equation

$$\sum_s \delta\left(p^2 - m^2\right)\left\{p^\mu \partial_{x\mu} f_s + \partial_{p\mu}\left[F^{\mu\nu}p_\nu + \frac{\hbar}{4}s\Sigma^{(0)\nu\rho}(\partial^\mu F_{\nu\rho})\right] f_s\right\} = 0.$$

- Force on particle: first Mathisson-Papapetrou-Dixon (MPD) equation
  $\rightarrow$ Particle with classical dipole moment $\Sigma^{(0)\mu\nu}$ in electromagnetic field:

W. Israel, General Relativity and Gravitation 9 (1978) 451

$$m\frac{d}{d\tau}p^\mu = F^{\mu\nu}p_\nu + \frac{\hbar}{4}s\Sigma^{(0)\nu\rho}(\partial^\mu F_{\nu\rho}).$$

$\tau$: worldline parameter, $\frac{d}{d\tau} = \dot{x}^\mu \frac{\partial}{\partial x^\mu} + \dot{p}^\mu \frac{\partial}{\partial p^\mu}$.

# Kinetic equation for dipole moment

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- $\bar{\Sigma}_{\mu\nu}$ determined by kinetic equation for dipole moment:

$$\delta(p^2 - m^2)\left[p \cdot \nabla \bar{\Sigma}^{\mu\nu} - \bar{\Sigma}^{\lambda\nu} F^{\mu}_{\ \lambda} + \bar{\Sigma}^{\lambda\mu} F^{\nu}_{\ \lambda} + \frac{1}{2}(\partial_{x\alpha} F^{\mu\nu})\partial_{p\alpha} V^{(0)}\right] = 0.$$

- To zeroth order:

$$m\frac{d}{d\tau}\Sigma^{(0)\mu\nu} = \Sigma^{(0)\lambda\nu} F^{\mu}_{\ \lambda} - \Sigma^{(0)\lambda\mu} F^{\nu}_{\ \lambda}.$$

- Recover second MPD equation for dipole-moment tensor $\Sigma^{(0)}_{\mu\nu}$!

  W. Israel, General Relativity and Gravitation 9 (1978) 451

- Equivalent to Bargmann-Michel-Telegdi (BMT) equation

  V. Bargmann, L. Michel, and V.L. Telegdi, PRL 2 (1959) 435

$$m\frac{d}{d\tau}n^{(0)\mu} = F^{\mu\nu} n^{(0)}_{\nu},$$

with classical spin vector

$$n^{(0)\mu} = -\frac{1}{2m}\epsilon^{\mu\nu\alpha\beta} p_{\nu} \Sigma^{(0)}_{\alpha\beta}.$$

## Massless limit

- Non-relativistic dipole-moment tensor connected to spin three-vector $n^k$:

$$\Sigma^{ij} = \epsilon^{ijk} n^k .$$

- For massive particles: define spin in rest frame.

  U. Heinz, PLB 144 (1984) 228

$$\Sigma^{\mu\nu} = -\frac{1}{m} \epsilon^{\mu\nu\alpha\beta} p_\alpha n_\beta .$$

- For massless particles: define spin in arbitrary frame with four-velocity $u^\mu$. Spin vector $n^\mu$ is always parallel to momentum.

  J-Y. Chen, D.T. Son, and M. Stephanov, PRL 115 (2015) 021601

$$\Sigma_u^{\mu\nu} = -\frac{1}{p \cdot u} \epsilon^{\mu\nu\alpha\beta} u_\alpha p_\beta .$$

- Massless limit: replace massive by massless dipole-moment tensor $\Sigma^{\mu\nu} \to \Sigma_u^{\mu\nu}$.
- Result agrees with previously known massless solution!

  Y. Hidaka, S. Pu, and D-L. Yang, PRD 95 (2017) 091901

  A. Huang, S. Shi, Y. Jiang, J. Liao, and P. Zhuang, PRD 98 (2018) 036010

  J.-H. Gao, Z.-T. Liang, Q. Wang, and X.-N. Wang, PRD 98 (2018) 036019

# Vector current in global equilibrium

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- In global equilibrium: Analytic solution for Boltzmann equation.
- Vector current is explicitly calculated as:

$$\mathcal{V}^\mu = \frac{2}{(2\pi\hbar)^3}\sum_s \Big[\delta(p^2-m^2)\Big(p^\mu - m\hbar\frac{s}{2}\tilde{\omega}^{\mu\nu}n_\nu^{(0)}\partial_{\beta\cdot\pi}\Big) + \hbar s\tilde{F}^{\mu\nu}n_\nu^{(0)}\delta'(p^2-m^2) + \hbar\frac{s}{2m}\delta(p^2-m^2)\epsilon^{\nu\mu\alpha\beta}p_\alpha\nabla_\nu n_\beta^{(0)}\Big] f_s^{(0)},$$

with zeroth-order equilibrium distribution function

$$f_s^{(0)} = [\exp(\beta\cdot\pi - \beta\mu_s)+1]^{-1},$$

where $\pi^\mu$ canonical momentum, $\beta^\mu$ thermal fluid velocity, $\beta$ inverse temperature, $\mu_s$ chemical potential.

- Analogue of chiral vortical effect (CVE) for massive particles.

  D. T. Son and P. Surowka, PRL 103 (2009) 0906.5044

- Analogue of chiral magnetic effect (CME).

  D. E. Kharzeev, L. D. McLerran, and H. J. Warringa, NPA 803 (2008) 0711.0950

- Thermal vorticity tensor: $\omega_{\mu\nu} \equiv \frac{1}{2}(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu)$.
- Dual thermal vorticity tensor: $\tilde{\omega}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\omega^{\alpha\beta}$.

# Axial-vector current in global equilibrium

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- Obtain expression for axial-vector current:

$$\mathcal{A}^\mu = \frac{2}{(2\pi\hbar)^3}\sum_s \Big[\delta(p^2-m^2)\Big(s\, m\, n^{(0)\mu} - \frac{\hbar}{2}\tilde{\omega}^{\mu\nu} p_\nu \partial_{\beta\cdot\pi}\Big) + \hbar\tilde{F}^{\mu\nu} p_\nu \delta'(p^2-m^2)\Big] f_s^{(0)} - \frac{\hbar}{2}\epsilon^{\mu\nu\alpha\beta} p_\nu \Xi_{\alpha\beta}\, \delta(p^2-m^2)\,.$$

- Classical spin precession.
- Analogue of axial chiral vortical effect (ACVE).
- Analogue of chiral separation effect (CSE).

D. E. Kharzeev, J. Liao, S. A. Voloshin, and G. Wang, Prog. Part. NP 88 (2016), 1511.04050

# Fluid-dynamical equations with spin I

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- Particle current:

$$J^\mu = \int d^4p\, \mathcal{V}^\mu .$$

Not parallel to the fluid velocity!

$$\partial_\mu J^\mu = 0 .$$

Conserved!

- Canonical energy-momentum tensor (matter part):

$$T^{\mu\nu}_{mat} = \int d^4p\, p^\nu \mathcal{V}^\mu .$$

Not symmetric!

$$\partial_\mu T^{\mu\nu}_{mat} = F^{\nu\mu} J_\mu .$$

Conserved in combination with electromagnetic and interaction part.

## Fluid-dynamical equations with spin II

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- Canonical spin tensor (matter part):

$$S^{\lambda,\mu\nu}_{mat} = -\frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\int d^4p\,\mathcal{A}_\rho,$$

$$\hbar\,\partial_\lambda S^{\lambda,\mu\nu}_{mat} = T^{\nu\mu}_{mat} - T^{\mu\nu}_{mat}.$$

Not conserved!
Spin angular momentum and orbital angular momentum are converted into one another.
$\rightarrow$ Consideration of spin leads to additional fluid-dynamical equation of motion.

W. Florkowski, B. Friman, A. Jaiswal, and E. Speranza, PRC 97 (2018) 041901;
W. Florkowski, B. Friman, A. Jaiswal, R. Ryblewski, and E. Speranza, PRD 97 (2017) 116017;
W. Florkowski, F. Becattini, and E. Speranza, APB 49 (2018) 1409;
F. Becattini, W. Florkowski, and E. Speranza, PLB 789 (2019) 419-425;
K. Hattori, M. Hongo, X.-G. Huang, M. Matsuo, H. Taya, PLB 795 (2019) 100-106

## Conclusions

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- Derived transport equation for distribution function and polarization for massive spin-1/2 particles in inhomogeneous electromagnetic fields.
- Recovered classical equations of motion.
- Solution agrees with previously known massless solution in massless limit.
- Derived explicit expressions for currents in global equilibrium.
- Found analogues of CVE, CME, ACVE, and CSE for massive particles.
- Derived fluid-dynamical equations of motion.

# Outlook

- Solve kinetic equations.
- Include collisions.
  → Boltzmann equation without assuming equilibrium.
  → Non-local collision term for spin-orbit interaction.
- Derive equations of motion for dissipative quantities.
  → Method of moments.

# Back-up

# Spin tensor vs. dipole-moment tensor

- Dipole-moment tensor:

$$s\Sigma^{\mu\nu} = \frac{1}{2m}\bar{u}_s \frac{i}{2}[\gamma^\mu, \gamma^\nu] u_s$$
$$= -s\frac{1}{m}\epsilon^{\mu\nu\alpha\beta} p_\alpha n_\beta$$

- Spin tensor:
  rank-3 tensor $S^{\lambda,\mu\nu}$ such that total angular momentum

$$J^{\lambda,\mu\nu} = x^\mu T^{\lambda\nu} - x^\mu T^{\lambda\mu} + \hbar S^{\lambda,\mu\nu}$$

# Diagonal spin basis

- Distribution function $f_{rs}$ is Hermitian matrix in spin space.
- Can be diagonalized by Unitary transformation:

$$f_{rs} = D_{rr'} \tilde{f}_{s'} \delta_{r's'} D^{\dagger}_{s's} .$$

- Redefine spinors

$$\tilde{u}_s \equiv \sum_{s'} u_{s'} D_{s's} .$$

- Define

$$sn^{\mu} \equiv \bar{\tilde{u}}_s \gamma^{\mu} \gamma^5 \tilde{u}_s .$$

- Only diagonal part contributes!

# Massless limit: details

- $p \cdot u$ related to rest-frame energy $\sqrt{p^2} \to \delta$-function!
- Massless limit: replace massive by massless dipole-moment tensor $\Sigma^{\mu\nu} \to \Sigma_u^{\mu\nu}$.
- Attention: $\delta(p^2 - m^2)/m \to \delta(p^2)/(p \cdot u)$.
- Find general solution for constraint on $\bar{\Sigma}^{\mu\nu}$.
- Define right- and left-handed currents $J^{\pm}_{\mu} \equiv \frac{1}{2}(\mathcal{V}^{m=0}_{\mu} \pm \mathcal{A}^{m=0}_{\mu})$
- Result

$$J^{\pm}_{\mu} = \left[ p_\mu \delta(p^2) \pm \frac{1}{2}\hbar \epsilon_{\mu\nu\alpha\beta} p^\nu F^{\alpha\beta} \delta'(p^2) \pm \frac{1}{2}\hbar \epsilon_{\mu\nu\alpha\beta} \frac{p^\alpha u^\beta}{p \cdot u} \delta(p^2) \nabla^\nu \right] f_{\pm} .$$

agrees with previously known massless solution!

Y. Hidaka, S. Pu, D-L. Yang, PRD 95 (2017) 091901; A. Huang, S. Shi, Y. Jiang, J. Liao, and P. Zhuang, PRD 98 (2018) 036010; J.-H. Gao, Z.-T. Liang, Q. Wang, and X.-N. Wang, PRD 98 (2018) 036019

# Distribution functions in global equilibrium

- Equilibrium distribution function:

$$f_s^{eq} = \left(e^{g_s} + 1\right)^{-1},$$

with $g$ linear combination of conserved quantities charge, momentum, and angular momentum:

$$g_s = \beta\pi \cdot U - \beta\mu_s + \frac{\hbar}{4} s\Sigma^{\mu\nu}\partial_\mu(\beta U_\nu).$$

Here, $\pi_\mu \equiv p_\mu + A_\mu$ is canonical momentum, $U_\mu$ is fluid velocity, $\beta \equiv \frac{1}{T}$ is inverse temperature, and $\mu_s$ is chemical potential.

- To zeroth order

$$f_s^{(0)} = \left(e^{g_{s0}} + 1\right)^{-1},$$

with

$$g_{s0} = \beta(\pi \cdot U - \mu_s).$$

# Equilibrium conditions

- "Homogeneous" part of the Boltzmann equation fulfilled if:

$$\begin{aligned} \mu_s &= \text{const}, \\ \partial_\nu \beta_\mu + \partial_\mu \beta_\nu &= 0, \end{aligned}$$

- "Inhomogeneous" part of Boltzmann equation:
  additional conditions to make global equilibrium possible, e.g.
  $\mu_{s=+} - \mu_{s=-} = 0$ or
  $\partial_{x\alpha} F_{\mu\nu} = 0$.