Wigner Function in Strong Electromagnetic Fields

Xin-li Sheng[1,2]

[1]Department of Modern Physics, University of Science and Technology of China
[2]Institute for Theoretical Physics, Goethe University

Fudan University, Shanghai, China,
October 30, 2019

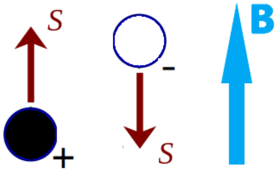
Outline

Landau levels

- Landau energy levels

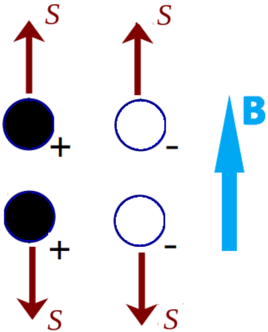
$$E^{(n)} = \sqrt{m^2 + p_z^2 + 2n|eB|}$$

- **Lowest Landau level**



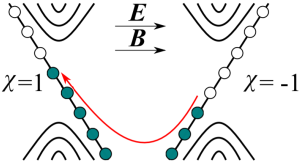
- Spin constrained by
  Principle of minimum energy

- **Higher Landau levels**



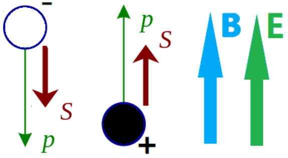
Axial-charge production

- **Electron pumping**



P. Hosur and X. Qi, 2013.

- **Axial-charge production**

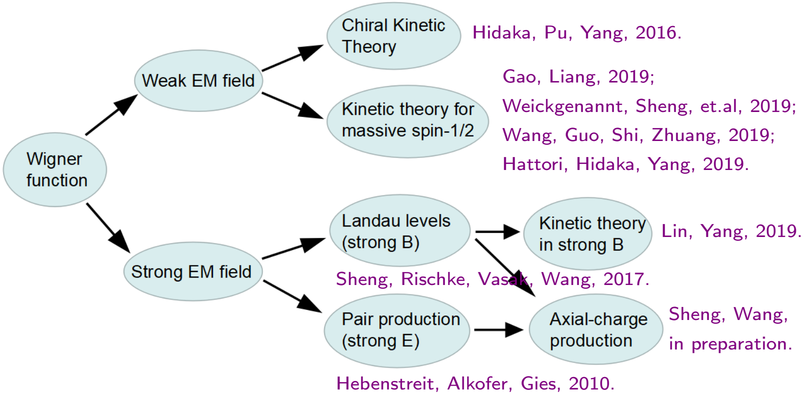


- Axial Ward identity

$$\partial_\mu j_5^\mu = -2\, m\, P - \frac{e^2}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu}$$

Fukushima, Kharzeev, Warringa, 2010.

Applications of Wigner function



∗ Motivation: Studying strong-field physics using Wigner function.

Wigner function: definition

- Covariant Wigner function

$$W(x,p) = \int \frac{d^4x'}{(2\pi)^4} e^{-\frac{i}{\hbar}p\cdot x'} U\left(x+\frac{x'}{2}, x-\frac{x'}{2}\right) \left\langle \bar{\psi}\left(x+\frac{x'}{2}\right) \otimes \psi\left(x-\frac{x'}{2}\right)\right\rangle .$$

- Equal-time Wigner function

$$W(t,\mathbf{x},\mathbf{p}) = \int dp^0 W(x,p).$$

- Not normal-ordered!

  $\Rightarrow$ Important for deriving the Schwinger pair production

- Expansion in terms of $\Gamma_i = \{\mathbb{I}_4,\ i\gamma^5,\ \gamma^\mu,\ \gamma^5\gamma^\mu,\ \frac{1}{2}\sigma^{\mu\nu}\}$ (generators of Clifford algebra),

$$W = \frac{1}{4}\left(\mathbb{I}_4\mathcal{F} + i\gamma^5\mathcal{P} + \gamma^\mu\mathcal{V}_\mu + \gamma^5\gamma^\mu\mathcal{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}\right).$$

Physical interpretations and analytical cases

- Expansion coefficients: real functions

	Property	Physical meaning (distribution in phase space)
$\mathcal{F}$	Scalar	Mass
$\mathcal{P}$	Pesudo-scalar	Pesudo-scalar condensate
$\mathcal{V}^{\mu}$	Vector	Net fermion current
$\mathcal{A}^{\mu}$	Axial-vector	Polarization (or spin current)
$\mathcal{S}^{\mu\nu}$	Tensor	Electric/magnetic dipole-moment

- Analytical solvable cases:
  free fermions; constant **E** [1]; constant **B** [2]; constant parallel **E** and **B** [3].
  1. Hebenstreit, Alkofer, Gies, 2010.
  2. Sheng, Rischke, Vasak, Wang, 2017; Gorbar, Miransky, Shovkovy, Sukhachov, 2017.
  3. Sheng, Fang, Wang, Rischke, 2018.
- Semi-classical method in weak fields $\Rightarrow$ Solutions up to linear order in $\hbar$
  $\Rightarrow$ See presentations of Nora Weickgenannt and Di-lun Yang.

Outline

Dirac equation

- Procedure for deriving Wigner function:
  Dirac equation ⇒ Eigen energies and wave functions
  ⇒ Quantized Dirac field ⇒ Wigner function
- Dirac equation with chiral imbalance:

$$\left[i\gamma^{\sigma}(\partial_{\sigma} + ieA_{\sigma}) - m + \mu\gamma^{0} + \mu_5\gamma^{0}\gamma_5\right]\psi(x) = 0$$

- Eigen energies

$$E_{p_z}^{(n)} = \pm\sqrt{m^2 + \left(\sqrt{p_z^2 + 2neB} \pm \mu_5\right)^2} - \mu$$

in massless limit ⇒

$$E_{p_z}^{(n)} = \pm\sqrt{p_z^2 + 2neB} - (\mu \pm \mu_5)$$

⋆ μ_5 represents chiral chemical potential in massless limit; describes spin imbalance in massive case.

CME and CSE

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

- Wigner function obtained analytically for constant **B** and constant μ_5.
  See more details in our paper: Sheng, Rischke, Vasak, Wang, 2017.
- Chiral Magnetic Effect:

$$j_z = \frac{eB}{4\pi^2}\int dp_z \frac{p_z-\mu_5}{E^{(0)}_{p_z}}\left[f_{FD}\left(E^{(0)}_{p_z}-\mu\right)+f_{FD}\left(E^{(0)}_{p_z}+\mu\right)-1\right]$$
$$= \frac{\mu_5}{2\pi^2}eB$$

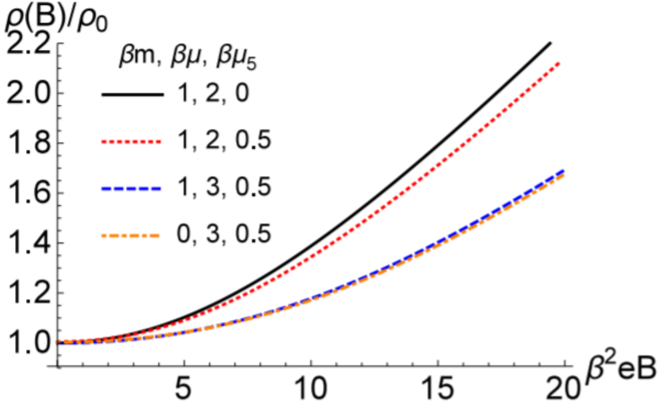
- Chiral Separation Effect:

$$j_{5z} = \frac{eB}{4\pi^2}\int dp_z\left[f_{FD}\left(E^{(0)}_{p_z}-\mu\right)+f_{FD}\left(E^{(0)}_{p_z}+\mu\right)\right]$$
$$= \frac{\mu}{2\pi^2}eB-\frac{\beta m^2}{4\pi^2}eB\int_0^\infty dp\,\frac{e^{\beta(p-\mu)}(e^{2\beta\mu}-1)(e^{2\beta p}-1)}{p\left[1+e^{\beta(p+\mu)}\right]^2\left[1+e^{\beta(p-\mu)}\right]^2}+\cdots$$

Fermion-number

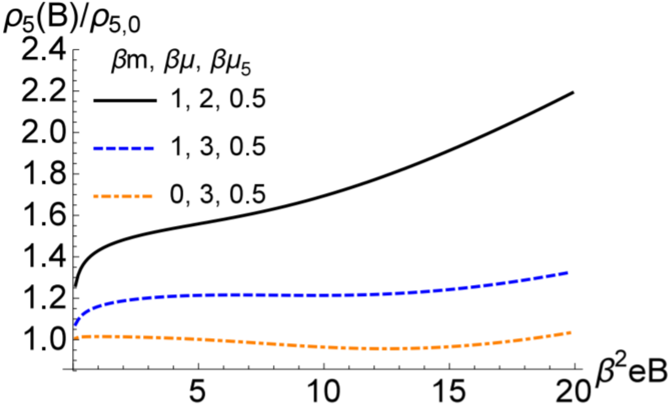
GOETHE UNIVERSITÄT FRANKFURT AM MAIN

- Fermion-number density as function of magnetic field strength



Axial charge

- Axial-charge density as function of magnetic field strength



Outline

Kinetic equation

- Kinetic equation:

$$D_t W = \frac{1}{2} \mathbf{D}_x \cdot \left[W, \gamma^0 \gamma\right] - i\, \mathbf{\Pi} \cdot \left\{W, \gamma^0 \gamma\right\} + i\, m \left[W, \gamma^0\right]$$

- After properly choosing a set of basis according to Landau levels,

$$(\partial_t + eE\partial_{p_z}) \begin{pmatrix} \chi_1^{(n)} \\ \chi_2^{(n)} \\ \chi_3^{(n)} \end{pmatrix} = 2 \begin{pmatrix} 0 & 0 & p_z \\ 0 & 0 & -m^{(n)} \\ -p_z & m^{(n)} & 0 \end{pmatrix} \begin{pmatrix} \chi_1^{(n)} \\ \chi_2^{(n)} \\ \chi_3^{(n)} \end{pmatrix}$$

- Asymptotic conditions:

$$\lim_{E \to 0} \begin{pmatrix} \chi_1^{(n)} \\ \chi_2^{(n)} \\ \chi_3^{(n)} \end{pmatrix} = \frac{1}{\sqrt{m^2 + p_z^2 + 2neB}} \begin{pmatrix} m^{(n)} \\ p_z \\ 0 \end{pmatrix}$$

with effective mass at n-th Landau level $m^{(n)} \equiv \sqrt{m^2 + 2neB}$.

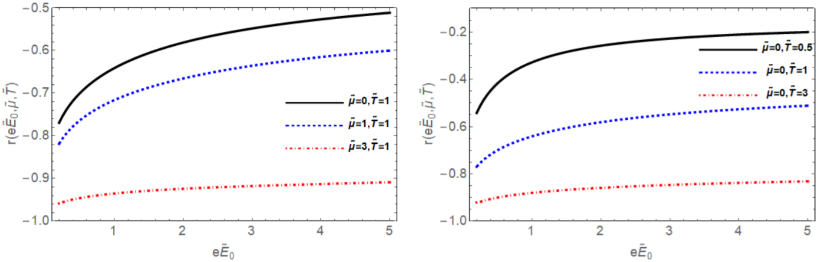
- Equivalent to Vlasov equation for pair production. Analytically solvable for constant $\mathbf{E}$. Sheng, Fang, Wang, Rischke, 2018.

Thermal suppression to pair production

- Pair production at n-th Landau level

$$\frac{d}{dt} n^{(n)}_{\mathrm{pair}} = \left(1 - \frac{\delta_{n0}}{2}\right) \frac{e^2 BE}{2\pi^2} \exp\left(-\pi \frac{m^2 + 2neB}{eE}\right) \left[1 + r\left(\frac{eE}{m^2}, \frac{T}{m}, \frac{\mu}{m}\right)\right]$$

- $r\left(\frac{eE}{m^2}, \frac{T}{m}, \frac{\mu}{m}\right)$ describes suppression from Pauli-exclusion principle to pair-production rate for a near-equilibrium system.



Axial-charge density and Pseudo-scalar condensate

- Axial-charge density, pseudo-scalar condensate

$$n_5(t) = 2\pi eB \int dp_z d_1\left(\frac{m^2}{eE}, \sqrt{\frac{2}{eE}}p_z\right) C\left(p_z - eEt\right)$$

$$P(t) = -2\pi eB \int dp_z \frac{m}{\sqrt{2eE}} d_3\left(\frac{m^2}{eE}, \sqrt{\frac{2}{eE}}p_z\right) C\left(p_z - eEt\right)$$

where

$$d_1(\eta, u) = -1 + e^{-\frac{\pi\eta}{4}}\eta \left|D_{-1-i\eta/2}(-ue^{i\frac{\pi}{4}})\right|^2,$$

$$d_3(\eta, u) = e^{-\frac{\pi\eta}{4}} e^{-i\frac{\pi}{4}} D_{-1-i\eta/2}(-ue^{i\frac{\pi}{4}}) D_{i\eta/2}(-ue^{-i\frac{\pi}{4}}) + c.c.,$$

- Function $C\left(p_z - eEt\right)$ includes distributions for particles and anti-particles,

$$C\left(p_z - eEt\right) = \frac{1}{(2\pi)^3}\left[f(p_z - eEt) + \bar{f}(p_z - eEt) - 1\right]$$

Axial Ward identity

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

Preliminary

- Separating vacuum part and medium part:

$$\frac{d}{dt} n_{5,\mathrm{vac}}(t) = \frac{e^2 BE}{2\pi^2} \exp\left(-\pi \frac{m^2}{eE}\right) - \frac{e^2 BE}{2\pi^2}$$

- The first term is renormalized axial-charge production rate.
- Dynamical equation for axial-charge density

$$\frac{d}{dt} n_{5,\mathrm{medium}}(t) + \frac{e^2 BE}{2\pi^2} \exp\left(-\pi \frac{m^2}{eE}\right) = -2\, m\, P(t) + \frac{e^2 BE}{2\pi^2}$$

- Extrapolate to Axial Ward Identity:

$$\partial_\mu j_5^\mu = -2\, m\, P - \frac{e^2}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu}$$

Outline

In weak-field limit

- Considering a system which is in thermal equilibrium at time $t = 0$.
- Focusing on $0 < t < \tau_{\text{collision}}$.
- Parameterize pseudo-scalar condensate as

$$P = \frac{e^2 BE}{2\pi^2 m}\left[1 - \exp\left(-\pi\frac{m^2}{eE}\right)\right]\left[1 - R\left(m,\, \mu,\, T,\, eE\right)\right]$$

- In weak-field limit

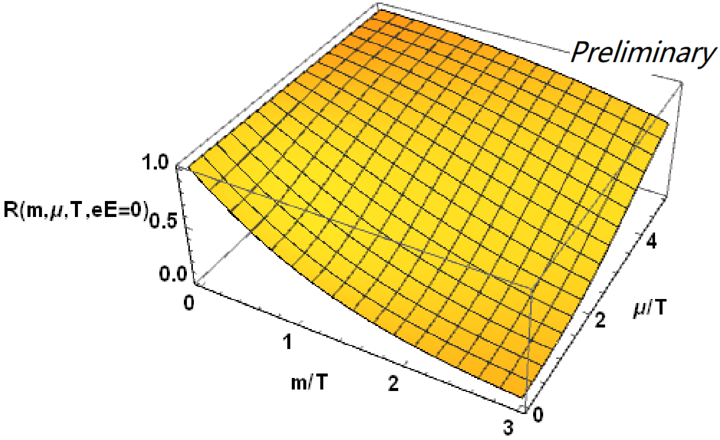
Preliminary

$$\lim_{eE\to 0} R\left(m,\, \mu,\, T,\, eE\right) = \int dp_z \frac{m^2}{2(m^2+p_z^2)^{3/2}} \times\left[\frac{1}{1+e^{\left(\sqrt{m^2+p_z^2}-\mu\right)/T}} + \frac{1}{1+e^{\left(\sqrt{m^2+p_z^2}+\mu\right)/T}}\right]$$

- Agrees with Copinger, Fukushima, Pu, 2018.

Medium correction

- Chemical potential and temperature dependence of medium correction

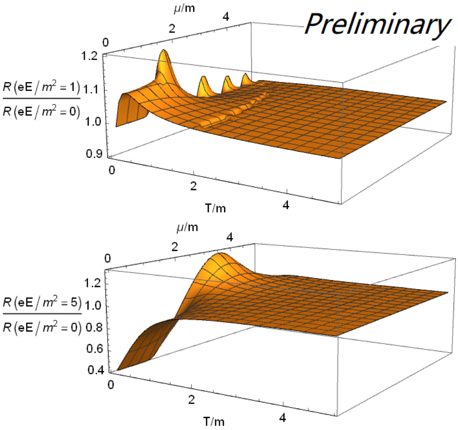


Stronger field

- Parameterize pseudo-scalar condensate as

$$P = \frac{e^2 BE}{2\pi^2 m}\left[1 - \exp\left(-\pi \frac{m^2}{eE}\right)\right] \times \left[1 - R\left(m,\ \mu,\ T,\ eE\right)\right]$$

- Depends on electric field strength E and thermodynamical quantities $\mu,\ T$.



Summary and outlook

- Derived Wigner function in constant **B**.
- Reproduced CME and CSE.
- Studied magnetic-field dependence of fermion number/ axial charge.
- Computed pair and axial charge production rates in constant **E** $\parallel$ **B**.
- Reproduced axial Ward identity.
- Computed pseudo-scalar condensate in a near-equilibrium system.

- Kinetic theory in strong background field; spin magnetohydrodynamics.