Quantum kinetic theory and spin polarization for Dirac fermions

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Introduction and motivations

From Boltzmann equation to quantum kinetic equation

- Classical kinetic theory: Boltzmann equation

$$(\partial_t + \dot{\mathbf{x}} \cdot \partial_{\mathbf{x}} + \dot{\mathbf{p}} \cdot \partial_{\mathbf{p}}) f = C[f]. \tag{1}$$

- EM field: Einstein-Vlasov equation

$$\delta(p^2 - m^2) p^\mu [\partial_\mu - F_{\mu\nu} \partial_p^\nu] f = 0. \tag{2}$$

- quantum kinetic theory: spin effect in $O(\hbar)$.
  - Chiral fermions: spin parallel to momentum. Berry curvature: $\frac{\hat{\mathbf{p}}}{2|\mathbf{p}|^2}$.
  - Massive fermions: New degrees of freedom for spin direction.
  - Spin evolution equation.

Ref:

Chiral kinetic theory: Stephanov, Yin. 2012; Son, Yamamoto. 2013; Hidaka, Pu, Yang. 2017; Huang, Shi, Jiang, Liao, Zhuang. 2018; Gao, Liang, Wang, Wang. 2018; Liu, Gao, Mameda, Huang. 2019.

Massive kinetic theory: Weickgenannt, Sheng, Speranza, Wang, Rischke. 2019; Gao, Liang. 2019; Hattori, Hidaka, Yang. 2019; Wang, Guo, Shi, Zhuang. 2019.

Introduction and motivations

Spin polarization

- Spin polarization is one of important probes in experimental physics to study the nuclear matter in heavy ion collisions.
- Spin polarization can be induced by vorticity $\boldsymbol{\omega}$ and magnetic field $\mathbf{B}$.(Liang, Wang. 2006; Becattini, Piccinini, Rizzo. 2008; Kharzeev, McLerran, Warringa. 2008.)
- Pauli-Lubanski vector with momentum $\hat{P}^C_\nu$ and spin operator $\hat{\mathcal{S}}^C_{\rho\sigma}$ (Ryder. QFT. 1996.)

$$\hat{\mathscr{W}}^\mu_C \equiv -\epsilon^{\mu\nu\rho\sigma}\hat{P}^C_\nu \hat{\mathcal{S}}^C_{\rho\sigma} \tag{3}$$

We can introduce the investigation of spin effects into nonequilibrium state via quantum kinetic theory.

Wigner function and quantum kinetic theory

Wigner operator in curved spacetime

- Wigner operator

$$\hat{W}_{\alpha\beta}(x,p) \equiv \int \frac{\sqrt{-g}d^4y}{(2\pi)^4} e^{-ip\cdot y/\hbar} \left[\bar{\psi}(x) e^{1/2 y\cdot \overleftarrow{D}}\right]_\beta \left[e^{-1/2 y\cdot D}\psi(x)\right]_\alpha. \quad (4)$$

Where the derivative $\overleftarrow{D}_\mu (D_\mu)$ acting to the left(right).

- We emphasize that x in equation (4) is the coordinate of point(P) in curved spacetime, and y is vector in the tangent space of point P, and p is vector in cotangent space of P.
- Horizontal lifted covariant derivatives (Winter. 1985; Calzetta, Habib, Hu. 1988; Fonarev. 1994)

$$D_\mu \equiv \nabla_\mu - \Gamma^\lambda_{\mu\nu} y^\nu \frac{\partial}{\partial y^\lambda} + \Gamma^\lambda_{\mu\nu} p_\lambda \frac{\partial}{\partial p_\nu} \underbrace{+\Gamma_\mu + \frac{i}{\hbar} A_\mu}_{\text{connection for spinor}}, \quad (5)$$

$$\overleftarrow{D}_\mu \equiv \overleftarrow{\nabla}_\mu - \frac{\overleftarrow{\partial}}{\partial y^\lambda} \Gamma^\lambda_{\mu\nu} y^\nu + \frac{\overleftarrow{\partial}}{\partial p_\nu} \Gamma^\lambda_{\mu\nu} p_\lambda \overbrace{-\Gamma_\mu - \frac{i}{\hbar} A_\mu}, \quad (6)$$

where ∇_μ is the usual covariant derivative operator, A_μ is gauge field, $\Gamma_\mu \equiv -\frac{i}{4}\omega^{ab}_\mu \sigma_{ab}$ is spin connection with $\sigma_{ab} = \frac{i}{2}[\gamma_a, \gamma_b]$ and ω^{ab}_μ the vierbein connection.

- Vierbein: $e^a = e^a_\mu \partial^\mu$.

Dynamic equation for Wigner function

Up to $O(\hbar^2)$ order

$$\left[\gamma^\mu\left(\Pi_\mu + \frac{i\hbar}{2}\Delta_\mu\right) - m\right]\hat{W} = \frac{i\hbar^2}{32}\gamma^\mu R_{\mu\alpha\rho\sigma}\left[\partial_p^\alpha \hat{W}, \sigma^{\mu\nu}\right] - \frac{\hbar^3}{8\times 4!}(\nabla_\beta R_{\mu\alpha\rho\sigma})\gamma^\mu\left[\partial_p^\alpha\partial_p^\beta \hat{W}, \sigma^{\rho\sigma}\right] \quad (7)$$

with

$$\begin{aligned}\Pi_\mu &= p_\mu - \frac{\hbar^2}{12}(\nabla_\rho F_{\mu\nu})\partial_p^\nu\partial_p^\rho + \frac{\hbar^2}{24}R^\rho{}_{\sigma\mu\nu}\partial_p^\sigma\partial_p^\nu p_\rho + \frac{\hbar^2}{4}R_{\mu\nu}\partial_p^\nu\,,\\ \Delta_\mu &= D_\mu - F_{\mu\lambda}\partial_p^\lambda - \frac{\hbar^2}{12}(\nabla_\rho R_{\mu\nu})\partial_p^\rho\partial_p^\nu - \frac{\hbar^2}{24}(\nabla_\lambda R^\rho{}_{\sigma\mu\nu})\partial_p^\nu\partial_p^\sigma\partial_p^\lambda p_\rho \\ &\quad + \frac{\hbar^2}{8}R^\rho{}_{\sigma\mu\nu}\partial_p^\nu\partial_p^\sigma D_\rho + \frac{\hbar^2}{24}(\nabla_\alpha\nabla_\beta F_{\mu\nu} + 2R^\rho{}_{\alpha\mu\nu}F_{\beta\rho})\partial_p^\nu\partial_p^\alpha\partial_p^\beta\,,\end{aligned} \quad (8)$$

where $R^\mu{}_{\nu\rho\sigma}$ is Riemann curvature and $R_{\mu\nu}$ is Ricci tensor.

Decomposition of Wigner function

$$W = \frac{1}{4}[\mathcal{F} + i\gamma^5\mathcal{P} + \gamma^\mu\mathcal{V}_\mu + \gamma^5\gamma^\mu\mathcal{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}]. \tag{9}$$

The constraints for the decomposed coefficients

$$\Delta_\mu\mathcal{V}^\mu = \frac{\hbar^2}{24}(\nabla_\eta R_{\mu\nu})\partial_p^\nu\partial_p^\eta\mathcal{V}^\mu, \qquad \hbar\Delta_\mu\mathcal{A}^\mu = -2m\mathcal{P}, \tag{10}$$

$$\Pi_\mu\mathcal{V}^\mu - m\mathcal{F} = \frac{\hbar^2}{8}R_{\mu\nu}\partial_p^\nu\mathcal{V}^\mu, \qquad \Pi_\mu\mathcal{A}^\mu = \frac{\hbar^2}{8}R_{\mu\nu}\partial_p^\nu\mathcal{A}^\mu, \tag{11}$$

$$\Pi_\mu\mathcal{F} - m\mathcal{V}_\mu = \frac{\hbar}{2}\Delta^\nu\mathcal{S}_{\nu\mu}, \qquad \Pi_\mu\mathcal{P} = -\frac{\hbar}{4}\epsilon_{\mu\nu\rho\sigma}\Delta^\nu\mathcal{S}^{\rho\sigma}, \tag{12}$$

$$\hbar\Delta_{[\mu}\mathcal{V}_{\nu]} - \epsilon_{\mu\nu\rho\sigma}\Pi^\rho\mathcal{A}^\sigma = m\mathcal{S}_{\mu\nu} - \frac{\hbar^2}{16}\epsilon_{\mu\nu\alpha\beta}R^{\alpha\beta\rho\sigma}\partial_\rho^p\mathcal{A}_\sigma, \tag{13}$$

$$\hbar\Delta_{[\mu}\mathcal{A}_{\nu]} - \epsilon_{\mu\nu\rho\sigma}\Pi^\rho\mathcal{V}^\sigma = -\frac{\hbar^2}{16}\epsilon_{\mu\nu\alpha\beta}R^{\alpha\beta\rho\sigma}\partial_\rho^p\mathcal{V}_\sigma, \tag{14}$$

$$\frac{\hbar}{2}\Delta_\mu\mathcal{F} - \Pi^\nu\mathcal{S}_{\mu\nu} = -\frac{\hbar^2}{16}R_{\mu\nu\rho\delta}\partial_p^\nu\mathcal{S}^{\rho\delta} - \frac{\hbar^2}{8}R^{\rho\nu}\partial_\nu^p\mathcal{S}_{\rho\mu}, \tag{15}$$

$$\frac{\hbar}{2}\Delta_\mu\mathcal{P} - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\Pi^\nu\mathcal{S}^{\rho\sigma} = m\mathcal{A}_\mu - \epsilon_{\mu\sigma\delta\lambda}\frac{\hbar^2}{8}R_\rho{}^{\sigma\lambda\nu}\partial_\nu^p\mathcal{S}^{\rho\delta}. \tag{16}$$

with $X_{[\mu}Y_{\nu]} \equiv \frac{1}{2}(X_\mu Y_\nu - X_\nu Y_\mu)$.

Solutions up to $O(\hbar)$

- $\mathcal{P}$, $\mathcal{F}$ and $\mathcal{S}^{\mu\nu}$ can be expressed by $\mathcal{V}^{\mu}$ and $\mathcal{A}^{\mu}$.
- In classical limit $\hbar \to 0$

$$\mathcal{V}^{\mu}_{(0)} = 4\pi p^{\mu} f^{(0)} \delta(p^2 - m^2), \tag{17}$$

$$\mathcal{A}^{\mu}_{(0)} = 4\pi \mathscr{A}^{\mu}_{(0)} \delta(p^2 - m^2), \tag{18}$$

with $p_{\mu}\mathscr{A}^{\mu}_{(0)}\delta(p^2 - m^2) = 0$.

- In $O(\hbar)$, we can write $\Delta_{\mu} = \nabla_{\mu} + (-F_{\mu\lambda} + \Gamma^{\nu}_{\mu\lambda} p_{\nu})\partial^{\lambda}_{p}$

$$\begin{aligned}\mathcal{V}^{\mu}_{(1)} &= 4\pi\hbar\Big\{\left(p^{\mu} f^{(1)} + \frac{1}{2p\cdot n}\epsilon^{\mu\nu\rho\sigma} n_{\nu}\Delta_{\rho}\mathscr{A}^{(0)}_{\sigma}\right)\delta(p^2 - m^2) \\ &\quad + \widetilde{F}^{\mu\nu}\left(\mathscr{A}^{(0)}_{\nu} - \frac{p\cdot\mathscr{A}^{(0)}}{p\cdot n} n_{\nu}\right)\delta'(p^2 - m^2)\Big\},\end{aligned} \tag{19}$$

$$\mathcal{A}^{\mu}_{(1)} = 4\pi\hbar\{\mathscr{A}^{\mu}_{(1)}\delta(p^2 - m^2) + \widetilde{F}^{\mu\nu} p_{\nu} f^{(0)}\delta'(p^2 - m^2)\}, \tag{20}$$

where n^{μ} is a unit timelike frame vector, and we have $p_{\mu}\mathscr{A}^{\mu}_{(1)}\delta(p^2 - m^2) = 0$.

- Rewrite $\mathscr{A}^{\mu}_{(0)} = \mathscr{A}^{\mu}_{(0)\perp} + p_{\mu} f^{(0)}_{5}$, where $p_{\mu}\mathscr{A}^{\mu}_{(0)\perp} = 0$.

The chiral case: $m = 0$

Solutions

$$\mathcal{R}^{\mu}/\mathcal{L}^{\mu} = 4\pi\Big\{ \left[p^{\mu} f_{R/L} \pm \hbar \Sigma_{n}^{\mu\nu} \Delta_{\nu} f_{R/L}\right] \delta(p^2) \pm \hbar \widetilde{F}^{\mu\nu} p_{\nu} f_{R/L} \delta'(p^2) \Big\}, \quad (21)$$

where $\mathcal{R}^{\mu}/\mathcal{L}^{\mu} \equiv \frac{1}{2}(\mathcal{V}^{\mu} \pm \mathcal{A}^{\mu})$, and $\Sigma_{n}^{\mu\nu} = \frac{1}{2p\cdot n}\epsilon^{\mu\nu\rho\sigma} p_{\rho} n_{\sigma}$ is the spin tensor for chiral fermion.

Chiral kinetic theory in curved spacetime

- Chiral magnetic effect
- Chiral vortical effect
- rotating frame

Dr. Kazuya Mameda's talk, Oct. 31.

The massive case: $m \neq 0$

The massive case $m \neq 0$

We have $f_5^{(0)}\delta(p^2 - m^2) = 0$ and $\mathcal{A}^{\mu}_{(0)} = 4\pi\mathscr{A}^{\mu}_{(0)\perp}\delta(p^2 - m^2)$.

Remove the frame vector n^μ from the kinetic theory:

- Redefinition of the scalar distribution

$$f^{(1)} \to f^{(1)} + \frac{1}{2m^2 p\cdot n}\epsilon^{\mu\nu\rho\sigma}p_\mu n_\nu \Delta_\rho \mathscr{A}^{(0)}_{\perp\sigma}, \tag{22}$$

  n^μ is removed from the kinetic theory.

- The redefinition of $f^{(1)}$ is equivalent to identifying the frame n^μ as the particle's rest frame $n^\mu = \frac{p^\mu}{m}$.

Solutions up to $O(\hbar)$

- Massive case $m \neq 0$, we define $m\theta^{\mu} f_A \equiv \mathscr{A}^{\mu}_{(0)\perp} + \hbar \mathscr{A}^{\mu}_{(1)\perp}$, with $p^{\mu}\theta_{\mu} = 0$:

$$\begin{aligned} \mathcal{V}^{\mu} &= 4\pi \Big\{ p^{\mu} f \delta(p^2 - m^2) + m\hbar \widetilde{F}^{\mu\nu}\theta_{\nu} f_A \delta'(p^2 - m^2) \\ &\quad + \frac{\hbar}{2m}\epsilon^{\mu\nu\rho\sigma} p_{\nu} \Delta_{\rho} (\theta_{\sigma} f_A)\, \delta(p^2 - m^2) \Big\}, \qquad (23) \\ \mathcal{A}^{\mu} &= 4\pi \{ m\theta^{\mu} f_A \delta(p^2 - m^2) + \hbar \widetilde{F}^{\mu\nu} p_{\nu} f \delta'(p^2 - m^2) \}, \qquad (24) \end{aligned}$$

where $\Sigma^{\mu\nu}_S = \frac{1}{2m}\epsilon^{\mu\nu\rho\sigma}\theta_{\rho} p_{\sigma}$ is the spin tensor for massive fermion.

- $\mathcal{P}$, $\mathcal{F}$ and $\mathcal{S}^{\mu\nu}$ can be expressed by $\mathcal{V}^{\mu}$ and $\mathcal{A}^{\mu}$.

Quantum kinetic theory for massive fermions

$$\Delta_\mu \mathcal{V}^\mu = 0, \qquad p\cdot\Delta\mathcal{A}_\mu = F_{\mu\nu}\mathcal{A}^\nu + \frac{\hbar}{2}\epsilon_{\mu\nu\rho\sigma}\Delta^\nu\Delta^\rho\mathcal{V}^\sigma. \tag{25}$$

- Two independent scalar kinetic equations

$$\begin{aligned}0 &= \delta(p^2 - m^2 \mp \hbar\Sigma_S^{\alpha\beta}F_{\alpha\beta}) \\ &\quad \times\left\{\left[p^\mu\Delta_\mu \pm \frac{\hbar}{2}\Sigma_S^{\mu\nu}\left(\nabla_\rho F_{\mu\nu}\partial_p^\rho + [D_\mu, D_\nu]\right)\right]f_{\uparrow/\downarrow}\right. \\ &\quad \left. + \frac{\hbar}{2}(f_\uparrow - f_\downarrow)\left(\nabla_\rho F_{\mu\nu}\partial_p^\rho + [D_\mu, D_\nu]\right)\Sigma_S^{\mu\nu}\right\}. \end{aligned} \tag{26}$$

where $f_{\uparrow/\downarrow} \equiv \frac{1}{2}(f \pm f_A)$.

- Spin evolution equation

$$\begin{aligned} & p\cdot\Delta\theta^\mu\delta(p^2 - m^2) \\ =\ & F^{\mu\nu}\theta_\nu\delta(p^2 - m^2) - \frac{1}{f_A}\theta^\mu\left(p\cdot\Delta f_A\right)\delta(p^2 - m^2) \\ & + \frac{\hbar}{2mf_A}\epsilon^{\mu\nu\rho\sigma}p_\sigma\Delta_\nu\Delta_\rho f\delta(p^2 - m^2). \end{aligned} \tag{27}$$

Spin polarization

Spin operator and frame vector

- In $O(\hbar)$ we have

$$4\pi\hbar(p\cdot n)f_5\Sigma_n^{\mu\nu}\delta(p^2) = \mathrm{Tr}\left(\frac{\hbar}{4}\{\sigma^{\mu\nu},\gamma^\lambda\}n_\lambda W(x,p)\right),$$
$$4\pi\hbar m f_A\Sigma_S^{\mu\nu}\delta(p^2-m^2) = \mathrm{Tr}\left(\frac{\hbar}{4}\{\sigma^{\mu\nu},\gamma^\lambda\}n_\lambda W(x,p)\right)\Bigg|_{n^\alpha=\frac{p^\alpha}{m}}. \tag{28}$$

- The spin current in Noether's theorem

$$\hat{\mathcal{S}}_C^{\lambda,\mu\nu} \equiv \frac{\hbar}{4}\bar{\psi}\{\sigma^{\mu\nu},\gamma^\lambda\}\psi \tag{29}$$

- Spin operator in field theory

$$\hat{\mathcal{S}}_C^{\mu\nu} \equiv \hat{\mathcal{S}}_C^{\lambda,\mu\nu} n_\lambda. \tag{30}$$

$$\mathcal{S}_C^{\mu\nu} \equiv \mathrm{Tr}\left(\frac{\hbar}{4}\{\sigma^{\mu\nu},\gamma^\lambda\}n_\lambda W(x,p)\right). \tag{31}$$

Spin polarization

- The Pauli-Lubanski vector

$$\begin{aligned} \mathscr{W}^{\mu}(x,p) &\equiv -\frac{1}{\hbar(p\cdot n)}\epsilon^{\mu\nu\rho\sigma}p_{\nu}\mathcal{S}^{C}_{\rho\sigma}, \\ \Lambda^{\mu}(x) &\equiv \int_{p}\mathscr{W}^{\mu}(x,p). \end{aligned} \tag{32}$$

- Massive fermions

$$\Lambda^{\mu}_{(m\neq 0)} = \pi\int_{p}\delta(p^2-m^2)\left(4m\theta^{\mu}f_A-\hbar\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\partial^{p}_{\nu}f\right). \tag{33}$$

- Massless fermions

$$\Lambda^{\mu}_{(m=0)} = \pi\int_{p}\delta(p^2)\left[4\left(p^{\mu}f_5+\hbar\Sigma^{\mu\nu}_{n}\Delta_{\nu}f\right)-\hbar\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\partial^{p}_{\nu}f\right]. \tag{34}$$

Equilibrium state for massive fermions

- $f^{eq}_{\uparrow/\downarrow} = n_F(g_{\uparrow/\downarrow})$ with $g_{\uparrow/\downarrow} = p\cdot\beta + \alpha_{\uparrow/\downarrow} \pm \hbar\Sigma^{\mu\nu}_S\omega_{\mu\nu}$

$$\begin{aligned} \nabla_\mu\beta_\nu + \nabla_\nu\beta_\mu &= 0, \qquad \nabla_{[\mu}\beta_{\nu]} - 2\omega_{\mu\nu} = 0, \\ \alpha_\uparrow = \alpha_\downarrow &= \alpha, \qquad \nabla_\mu\alpha = F_{\mu\nu}\beta^\nu. \end{aligned} \tag{35}$$

- The spin per particle in phase space $\pi^\mu = \mathscr{W}^\mu/f$ induced by vorticity (Becattini, Chandra, Zanna, Grossi. 2013)

$$\pi^\mu_{\omega-eq} = 4\pi\hbar\epsilon^{\mu\sigma\alpha\beta}p_\sigma\nabla_\alpha\beta_\beta[1 - n_F(p\cdot\beta+\alpha)]\delta(p^2 - m^2). \tag{36}$$

- Spin polarization density

$$\Lambda^\mu_{eq} = -\pi\hbar\int_p \delta(p^2 - m^2) f'_{eq}\left(\epsilon^{\mu\nu\rho\sigma}p_\nu\nabla_\rho\beta_\sigma + \epsilon^{\mu\nu\rho\sigma}\beta_\nu F_{\rho\sigma}\right). \tag{37}$$

Equilibrium state for chiral fermions

- $f^{eq}_{R/L} = n_F(g_{R/L})$, with $g_{R/L} = p\cdot\beta + \alpha_{R/L} \pm \hbar\Sigma^{\mu\nu}_n \omega_{\mu\nu}$

$$\nabla_\mu\beta_\nu + \nabla_\nu\beta_\mu = \phi(x) g_{\mu\nu}, \qquad \nabla_{[\mu}\beta_{\nu]} - 2\omega_{\mu\nu} = 0 \tag{38}$$

$$\nabla_\mu\alpha_R = \nabla_\mu\alpha_L = F_{\mu\nu}\beta^\nu. \tag{39}$$

- Spin polarization density

$$\Lambda^\mu_{eq} = \pi\int_p \delta(p^2) f'_{eq}\big[2p^\mu(\alpha_R - \alpha_L) - \hbar\epsilon^{\mu\nu\rho\sigma} p_\nu \nabla_\rho\beta_\sigma - \hbar\epsilon^{\mu\nu\rho\sigma}\beta_\nu F_{\rho\sigma}\big]. \tag{40}$$

Summary and outlook

Summary

- We have derived covariant kinetic theory up to $O(\hbar)$ order in curved spacetime.
- For massive fermions, the frame vector can be removed in kinetic theory.
- Spin polarization is derived from kinetic theory, and the results are available in non-equilibrium state.

Outlook

- Quantum correction for collision term.
- Simulation of the evolution of spin polarization for Dirac fermions.
- From quantum kinetic theory to spin hydrodynamics.

Thank you!