A Generalized Fluctuation-Dissipation Theorem in Stochastic Hydrodynamics and Its Application

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New development of hydrodynamics and its applications in Heavy-Ion Collisions@Fudan University

Outline

- Motivations
  - to understand hydro matter in heavy-ion collisions
- Standard path integral representation of stochastic hydro
- Naive Einstein relation
- Application with multiplicative noises

Fluid Dynamics

Fluid dynamics is a universal EFT of non-equilibrium many body systems, including relativistic QCD matter, with a stable equation of state and

- Conservation of energy and momentum: $\partial_\mu T^{\mu\nu} = 0$
- Conservation of charge: $\partial_\mu J^\mu = 0$

The dissipation terms are described by

$$\pi^{ij} = -\eta\left(\partial^i u^j + \partial^j u^i - \frac{2}{3}\delta^{ij}\nabla\cdot\mathbf{u}\right) - \zeta\delta^{ij}\nabla\cdot\mathbf{u}$$

$$\nu^\mu = -\sigma T\Delta^{\mu\nu}\partial_\nu\left(\frac{\mu}{T}\right) \qquad \Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$$

for shear viscosity η, bulk viscosity ζ and charge conductivity σ.

Fluctuation in Hydro

- In the deterministic hydro equations, no spontaneous fluctuation arise
- The microscopic dynamics entails the occurrence of fluctuations, which has to persist on the coarse grained hydrodynamic level

$$
\begin{aligned}
T^{\mu\nu} &= \varepsilon\, u^\mu u^\nu + p\Delta^{\mu\nu} + \pi^{\mu\nu} + \xi^{\mu\nu} \\
J^\mu &= n\, u^\mu + \nu^\mu + \xi^\mu
\end{aligned}
$$

To figure out a low-energy effective hydro theory, need both dissipation (transport coefficients) and fluctuations (noises)

$$\langle \xi^{\mu\nu} \rangle = 0 \quad \langle \xi^{\mu} \rangle = 0$$

$$\left\langle (\xi^{\mu\nu})^2 \right\rangle \neq 0 \quad \left\langle (\xi^{\mu})^2 \right\rangle \neq 0$$

Stochastic Hydro Encoded in Noises

Firstly, coarse-grained treating for fast modes $\longrightarrow$ random noise $\xi(x,t)$

Secondly, mesoscopic equation of motion in Langevin way for slow variables $\longrightarrow$ $\psi(x,t)$

Obtaining stochastic hydro equations in terms of $\psi = (\delta n, \delta\varepsilon, \delta\pi_k)$:

$$
\begin{aligned}
\frac{\partial \delta n(x,t)}{\partial t} &= w\nabla^i\,\delta\pi_i(x,t) + \frac{\delta\pi^i\partial_i n}{h} + D_\sigma(x,t)\nabla^2\delta n(x,t) + \sqrt{g_n}\cdot\nabla\xi_n(x,t) + \sqrt{g_\varepsilon}\cdot\nabla\xi_\varepsilon(x,t) \\
\frac{\partial \delta\varepsilon(x,t)}{\partial t} &= \nabla^i\,\delta\pi_i(x,t) + \frac{\delta\pi^i\partial_i\varepsilon}{h} + \sqrt{g_n}\cdot\nabla\xi_n(x,t) \\
\frac{\partial \delta\pi_k(x,t)}{\partial t} &= \partial_k p + \frac{\pi^i\partial_i\pi_k}{h} + \gamma_s(x,t)\left(\nabla^2\delta_{kj} + \frac{1}{3}\partial_k\partial_j\right)\delta\pi^j(x,t) + \sqrt{g_k}\cdot\nabla\xi_k(x,t) \\
& h = \varepsilon + p \quad w = \frac{n}{h} \quad D_\sigma \propto \sigma \quad \gamma_s \propto \eta
\end{aligned}
$$

MSRJD Field Theory Representation for Langevin Equations

In terms of $\psi = (\delta n, \delta\varepsilon, \delta\pi_{\shortparallel}, \delta\pi_{\perp})$, the description of field theory write with auxiliary field $\tilde{\psi}$ as:

$$\begin{aligned}
\langle \mathcal{O}[\psi] \rangle &= \int \mathcal{D}\psi\, P[\xi]\mathcal{O}[\psi]\delta\left(\dot{\psi} - V[\psi] - F\nabla^2[\psi] - \mathcal{A}^{\frac{1}{2}} \cdot \nabla\xi\right) \\
&\quad \cdot \det\left(\frac{\delta\left(\dot{\psi} - V[\psi] - F\nabla^2[\psi] - \mathcal{A}^{\frac{1}{2}} \cdot \nabla\xi\right)}{\delta\psi}\right) \\
&= \mathcal{N}\int \mathcal{D}\psi\mathcal{D}\tilde{\psi}\, P[\xi]\mathcal{O}[\psi]\exp\left\{-\tilde{\psi}\left(\dot{\psi} - V[\psi] - F\nabla^2[\psi] - \mathcal{A}^{\frac{1}{2}} \cdot \nabla\xi\right)\right\} + \text{det term} \\
&= \mathcal{N}'\int \mathcal{D}\psi\mathcal{D}\tilde{\psi}\, \mathcal{O}[\psi]\exp\underbrace{\left\{-\tilde{\psi}\left(\dot{\psi} - V[\psi] - F\nabla^2[\psi]\right) - \tilde{\psi}\mathcal{A}\nabla^2\tilde{\psi}\right\}}_{\mathcal{S}[\psi,\tilde{\psi}],\ \text{desired effective action}} + \text{det term}
\end{aligned}$$

We get an effective action at low energies and near thermal equilibrium

Time Reversal and Action Symmetry

$$\mathcal{T}: \ \psi(T-t) = \epsilon\psi(t) \qquad \mathcal{T}: \ \tilde{\psi}(T-t) = \epsilon\tilde{\psi}(T-t) + \epsilon\left(\mathcal{A}\nabla^2\right)^{-1}\left(\dot{\psi} - V[\psi]\right) \qquad \epsilon = \pm 1$$

$\mathcal{T}$ is the time reversal operation $t \to T - t$

$$\begin{aligned}
\mathcal{T}: \mathcal{S}\left[\psi, \tilde{\psi}\right] + h.c. &= -\tilde{\psi}(-t)\left(-\dot{\psi} + V[\psi] - F\nabla^2[\psi]\right) - \tilde{\psi}(-t)\mathcal{A}\nabla^2\tilde{\psi}(-t) \\
&= -\left(\mathcal{T}\tilde{\psi}\right)\left(-\dot{\psi} + V[\psi] - F\nabla^2[\psi]\right) - \left(\mathcal{T}\tilde{\psi}\right)\mathcal{A}\nabla^2\left(\mathcal{T}\tilde{\psi}\right) \\
&= \mathcal{S}\left[\psi, \tilde{\psi}\right](-t) - F[\psi]\mathcal{A}^{-1}\left(\dot{\psi} - V[\psi]\right) + h.c.
\end{aligned}$$

If requires:

$$F^{\alpha}[\psi]\mathcal{A}^{-1}_{\alpha\beta} + h.c. = \beta\frac{\partial H}{\partial\psi_\beta}$$

$$\Longrightarrow \mathcal{A}_{\alpha\beta} = TF_{\alpha}\left(\frac{\partial H}{\partial\psi_\beta}\right)^{-1} + h.c.$$

Statement of Detailed Balance

$$F^{\alpha}[\psi]\mathcal{A}^{-1}_{\alpha\beta}V^{\beta}[\psi] + h.c. = \beta Q^{\beta\gamma}\frac{\partial H}{\partial\psi_\beta}\frac{\partial H}{\partial\psi_\gamma} = 0$$

where $V^\beta = Q^{\beta\gamma}\partial H/\partial\psi_\gamma$ and Q is the asymmetric matrix of Poisson bracket as definition. It is exactly know as the reversible stationary probability current being divergence-free.

The well-known stationary distribution $P_{st} \propto e^{-\beta H(x)}$: $\downdownarrows$

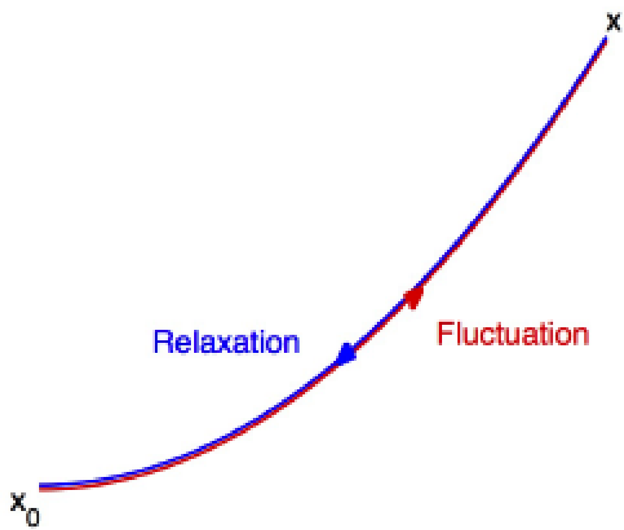
$$\int_0^T F_\alpha[\psi]\mathcal{A}^{-1}\dot{\psi}^\alpha d\tau + h.c. = -\ln P_{st}(T) + \ln P_{st}(0)$$

$$\Rightarrow \underbrace{e^{-S(x_T,T;x_0,0)}}_{w(x_0\to x_T)}\cdot P_{st}(x_0) = \underbrace{e^{-S(x_0,T;x_T,0)}}_{w(x_T\to x_0)}\cdot P_{st}(x_T)$$

☞ The fluctuation paths of the direct dynamics are the reversed of the relaxation paths in the language of dual dynamics, and vice versa.

Time Reversed Relaxation Paths Minimize the Effective Action

c.f. (F. Bouchet, J. Laurie and O. Zaboronski, J. Stat. Phys. 156, (2014) 1066)



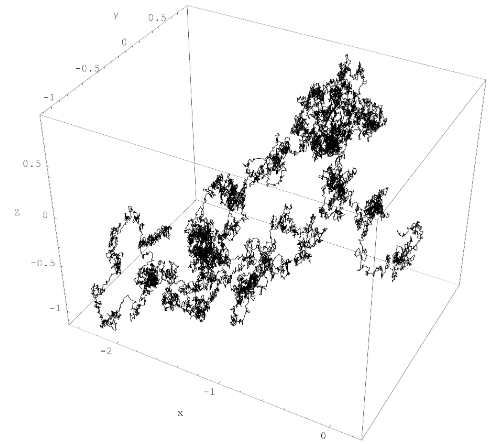
The minimizer of the action from an attractor of the system to any point of its basin of attraction is the reversed of the relaxation path.

This is an extended Onsager-Machlup relation. For time reversible systems, the most probable path to reach a state x (a fluctuation) is the time reversal of a relaxation path starting from x (dissipation).

The time reversed relaxation paths also minimizes the action like the fluctuations. The full action is completed in a conjugated dynamics way.

Recall of Einstein Relation

Take the Brownian motion $\dot{\vec{v}} + \gamma \vec{v} = \vec{\xi}(t)$ giving



$$\langle v_i(t_1) v_j(t_2) \rangle = \delta_{ij} \frac{D}{2\gamma} e^{-\gamma |t_1 - t_2|}$$

for $t_{1,2} \gg \gamma$, where $\langle \xi_i(t_1) \xi_j(t_2) \rangle = D\delta(t_1 - t_2)$.

What determines the noise strength D?

Assume the Brownian particle eventually equilibrates with the fluid at temperature $T = \left\langle v^2(t \to \infty) \right\rangle$ by averaging over ξ

The correlation functions satisfy

$$D = 2\gamma T$$

An old fashioned fluctuation-dissipation theorem!

Hamiltonian of Fluid

The Hamiltonian $\Delta\mathcal{H}$ of grand canonical ensemble of fluid system is related to the pressure via $\ln\Xi = pV/T = -\beta\Delta\mathcal{H}$.

$$p(x,t) = p(n(x,t),\varepsilon(x,t)) = \beta\varepsilon - \frac{\mu}{T}n - s$$

Energy and particle number defined for arbitrary system:

$$\varepsilon = u_\mu T^{\mu\nu}u_\nu \quad \text{and} \quad n = J^\mu u_\mu$$

Apply equilibrium EoS: $p = p_0(\varepsilon, n) \quad \text{and} \quad s = s_0(\varepsilon, n)$

$$-\beta\Delta\mathcal{H} = \Delta s = \frac{1}{2}\frac{\partial^2 s}{\partial n^2}(\delta n)^2 + \frac{\partial^2 s}{\partial n\partial\varepsilon}\delta n\delta\varepsilon + \frac{1}{2}\frac{\partial^2 s}{\partial\varepsilon^2}(\delta\varepsilon)^2$$

A Rigorous Fluctuation-Dissipation Theorem

The matrix of Hamiltonian is expressed as

$$\mathscr{H} = \begin{pmatrix} \theta_{nn} & \theta_{n\varepsilon} & 0 \\ \theta_{n\varepsilon} & \theta_{\varepsilon\varepsilon} & 0 \\ 0 & 0 & \frac{1}{h}\delta_{ij} \end{pmatrix}$$

Even though $\mathscr{F}_0 = \mathbf{diag}\,(D_\sigma, 0, \gamma_s)$, it gives

$$\mathscr{A}_0 = \mathscr{A}_0^{\mathrm{T}} = T \begin{pmatrix} \theta_{nn} D_\sigma & \theta_{n\varepsilon} D_\sigma & 0 \\ \theta_{n\varepsilon} D_\sigma & 0 & 0 \\ 0 & 0 & \frac{\gamma_s}{h}\delta_{ij} \end{pmatrix}$$

producing off diagonal noise terms, which originate in the coupled thermodynamical relation between ε and n.

E.o.S. is important because the mixed term $\propto \epsilon_{\varepsilon n} = \frac{\theta_{n\varepsilon}}{\theta_{nn}}$.

EFT in Fluid

Let $\Psi = \left(\psi, \tilde{\psi}\right)$, $\mathcal{L} = \Psi^T S \Psi$ and

$$S = \begin{pmatrix} 0 & \frac{\partial}{\partial t} - \mathcal{V} - \mathcal{F}\nabla^2 \\ -\frac{\partial}{\partial t} - \mathcal{V}^\dagger - \mathcal{F}^\dagger \nabla^2 & \mathcal{A}\nabla^2 \end{pmatrix}$$

with the inverse of the harmonic coupling matrix

$$S_0^{-1} = \begin{pmatrix} G_{\psi\psi} & G_{\psi\tilde{\psi}} \\ G^\dagger_{\psi\tilde{\psi}} & 0 \end{pmatrix}.$$

$$G_{\psi\psi} = G_{\psi\tilde{\psi}} \mathcal{A}_0 \mathbf{k}^2 G^\dagger_{\psi\tilde{\psi}}$$

The leaving interaction terms are

$$\mathcal{L}_I = \tilde{\psi} V_I[\psi] + \tilde{\psi} F_I \nabla^2[\psi] + \tilde{\psi}\mathcal{A}_I \nabla^2 \tilde{\psi} + h.c.$$

Multiplicative Noises

Expanding $\mathcal{F}_I[\psi]$ to triplet fields:

$$\mathcal{F}_I = \begin{pmatrix} \lambda_1\delta n + \lambda_2\delta\varepsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda_3\delta n \end{pmatrix}$$

$$\mathcal{A}_I = \begin{pmatrix} \lambda_1\delta n + \lambda_2\delta\varepsilon & \epsilon_{\varepsilon n}\left(\lambda_1\delta n + \lambda_2\delta\varepsilon\right) & 0 \\ \epsilon_{\varepsilon n}\left(\lambda_1\delta n + \lambda_2\delta\varepsilon\right) & 0 & 0 \\ 0 & 0 & \lambda_3\delta n \end{pmatrix}$$

Combing the TRS: $\tilde{\psi}(-t) = \epsilon\tilde{\psi}(-t) + \epsilon\left(\mathcal{A}\nabla^2\right)^{-1}\left(\dot{\psi} - V[\psi]\right)$ of detailed balance, causality and obvious $\langle V[\psi]\psi\rangle = 0$, the FDT manifests as:

$$\left\langle \psi(t_1)\mathcal{A}\nabla^2\tilde{\psi}(t_2)\right\rangle = \Theta(t_2, t_1)\left\langle \psi(t_1)\left(\dot{\psi}(t_2) - V[\psi](t_2)\right)\right\rangle$$

$$\Rightarrow \left\langle \psi_\alpha\dot{\psi}_\beta\right\rangle = \left\langle \psi_\alpha\left(\mathcal{A}_0\right)^\gamma_\beta\nabla^2\tilde{\psi}_\gamma\right\rangle \Rightarrow G_{\psi_\alpha\psi_\beta} = \frac{k^2}{\omega}\left(\mathcal{A}_0\right)^\gamma_\beta \,\mathbf{Im}\, G_{\psi_\alpha\tilde{\psi}_\gamma}$$

The well known representation of FDT is recovered for constant matrix $\mathcal{A}_0 = 2\mathcal{F}_0 T$

Jacobian Term and the Three Point Correlation

c.f. (Täuber, U. C. *Critical dynamics: A Field Theory Approach to Equilibrium and Non-Equilibrium Scaling Behavior*, Cambridge University Press, 2014.)

Let us now investigate the subset of diagrams stemming from the non-linear contribution in $-\int \mathrm{d}t \sum_\alpha \widetilde{\psi}^\alpha F^\alpha[\psi]$ of the response functional, which contain closed response loops, see e.g. Fig. 4.2(a) and (b). These must involve a contraction of an internal field ψ^β with the response field $\widetilde{\psi}^\alpha$, wherefrom we see that the response loop terms can be formally written as an effective vertex

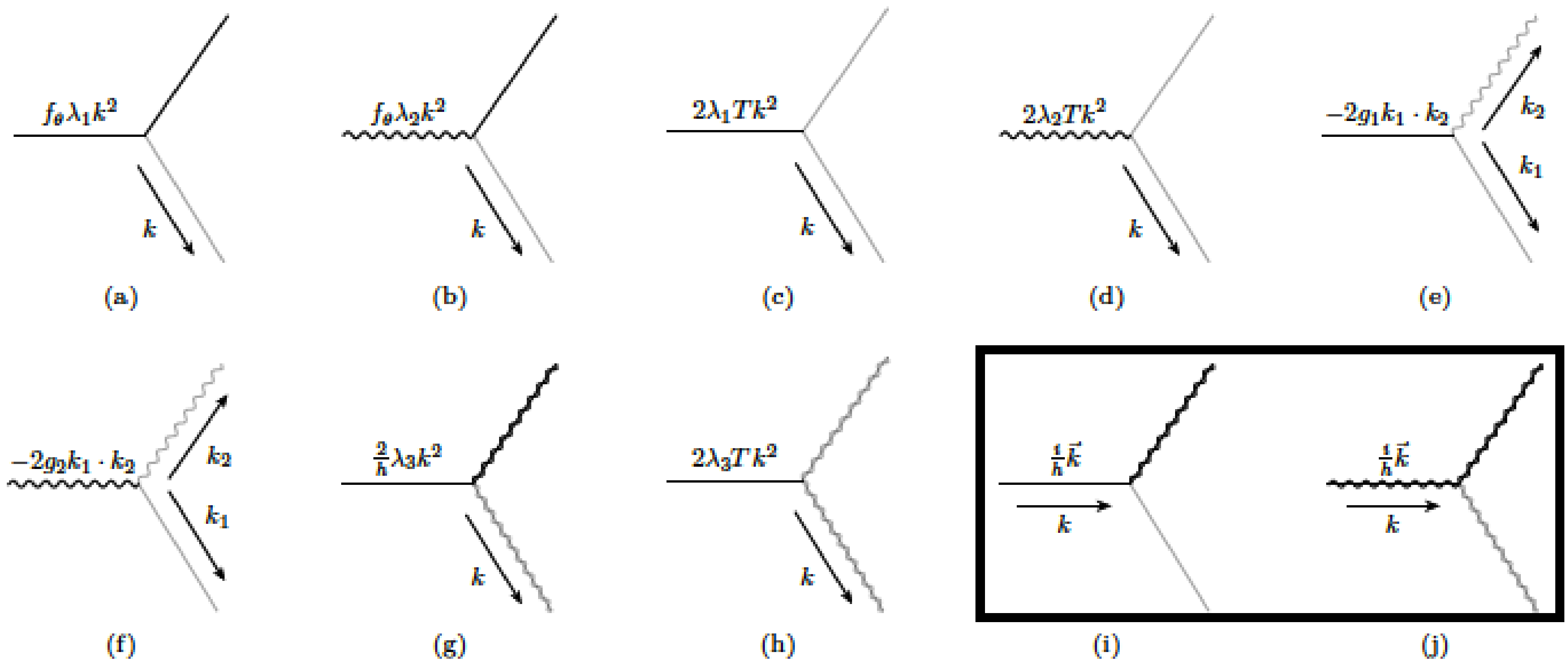
$$-\int \mathrm{d}^d x \int \mathrm{d}t \sum_{\alpha,\beta} \langle \widetilde{\psi}^\alpha \psi^\beta \rangle \frac{\delta F^\alpha[\psi]}{\delta \psi^\beta(x,t)} = -\int \mathrm{d}^d x \int \mathrm{d}t \sum_\alpha \Theta(0) \frac{\delta F^\alpha[\psi]}{\delta \psi^\alpha(x,t)} ,$$

where we have used Eq. (3.29). Consequently the terms from those Feynman diagrams that contain closed response loops are exactly canceled by the contributions from the functional determinant. As we have just established, this holds true for any choice of the discretization splitting, provided $\Theta(0) = \kappa$ is set consistently.

$$\left\langle \psi(x_1) \mathcal{A}_I \psi(x_2) \tilde{\psi}(x_2) \right\rangle = \left\langle \psi(x_1) \mathcal{A}_I \psi(x_2) \tilde{\psi}(x_2) \right\rangle_0 + i \left\langle \psi(x_1) \mathcal{A}_I \psi(x_2) \tilde{\psi}(x_2) S_I \right\rangle + \ldots\ldots$$

Feynman Diagrams

Including all the vertex terms of F_I, G_I and $V_I = \mathrm{Diag}\,(\pi_{||}\partial^x n, \pi_{||}\partial^x \varepsilon, \pi_{||}\partial^x \pi_{||})$.



Corrections of Propagators

For $\Lambda^2 < \omega < \Lambda$, all one loop contributions are cutoff dependent, which can be treated as renormalizing transport coefficients.

For $\omega < \Lambda^2$, the nontrivial results represented by I_1, J_1 and $L_{1,2,3}$ (seen details in backup slide). It leaves us

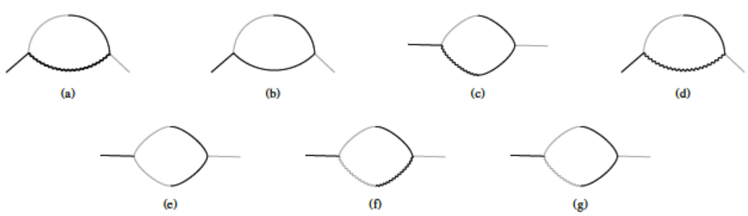


Figure 3: Denominator corrections of propagator *nn*

$Fig.(3a) \sim O\left(\omega^{\frac{3}{2}}\right)$ $Fig.(3b) \sim \lambda_1^2 I_1$

$Fig.(3c) \sim \lambda_1\lambda_2 I_1$ $Fig.(3d) \sim \lambda_2^2 I_1$

$Fig.(3e) \sim \lambda_1^2 T I_1$ $Fig.(3f) \sim \#\Lambda$ term

$Fig.(3g) \sim \epsilon_{\varepsilon n}\lambda_1^2 T I_1$

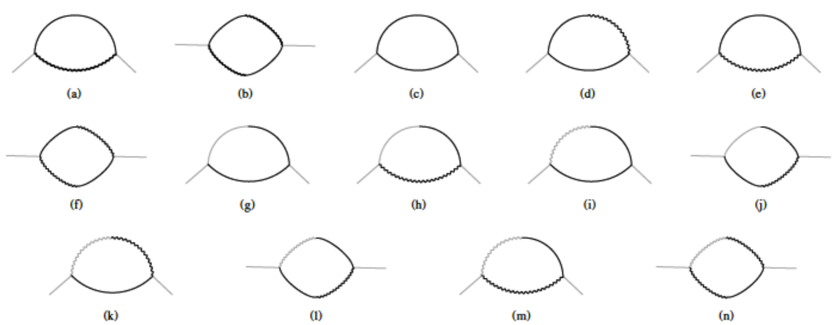


Figure 4: Numerator corrections of propagator *nn*

$Fig(4a,4b) \sim O\left(\omega^{\frac{3}{2}}\right)$ $Fig.(4c) \sim \lambda_1^2 J_1$

$Fig.(4d) \sim \lambda_1\lambda_2 J_1$ $Fig.(4e) \sim \lambda_2^2 J_1$

$Fig.(4f) \sim \lambda_2^2 L_3$ $Fig.(4g) \sim \lambda_1^2 T I_1$

$Fig.(4h) \sim \lambda_2^2 T I_1$ $Fig.(4i) \sim \epsilon_{\varepsilon n}\lambda_1^2 T I_1$

$Fig.(4j) \sim \lambda_1\lambda_2 T I_1$ $Fig.(4k) \sim \#\Lambda$ term

$Fig.(4l) \sim \epsilon_{\varepsilon n}\lambda_1\lambda_2 T L_1$

$Fig.(4m) \sim \epsilon_{\varepsilon n}\lambda_2^2 T L_2$ $Fig.(4n) \sim O\left(\omega^{\frac{3}{2}}\right)$

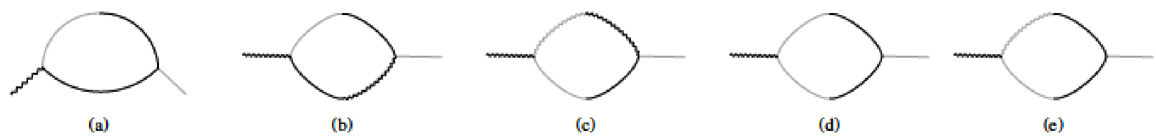


Figure 5: Denominator corrections of propagator ϵn

Figure 6: Numerator corrections of propagator ϵn

$Fig.\ (5a) \sim \lambda_1\lambda_2 I_1$ $\quad$ $Fig.\ (5b) \sim \lambda_2^2 I_1$ $\quad$ $Fig.\ (5c) \sim \#\Lambda$ term $\quad$ $Fig.\ (5d) \sim \lambda_1\lambda_2 T I_1$ $\quad$ $Fig.\ (5e) \sim \epsilon_{\varepsilon n}\lambda_1\lambda_2 T I_1$

$Fig.\ (6a) \sim \epsilon_{\varepsilon n}\lambda_2^2 T I_1$ $\quad$ $Fig.\ (6b) \sim \epsilon_{\varepsilon n}\lambda_1\lambda_2 T I_1$ $\quad$ $Fig.\ (6c) \sim \epsilon_{\varepsilon n}\lambda_1^2 T I_1$ $\quad$ $Fig.\ (6d) \sim \epsilon_{\varepsilon n}\lambda_1\lambda_2 T I_1$

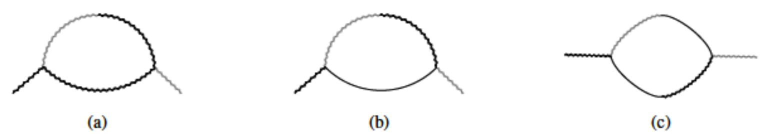


Figure 7: Denominator corrections of propagator $\pi_x\tilde{\pi}_x$

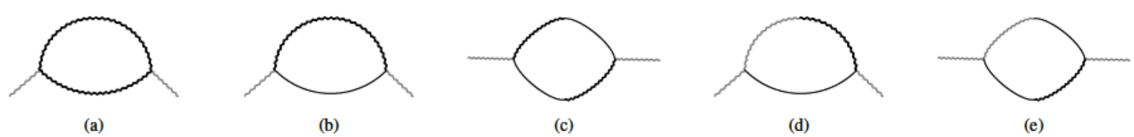


Figure 8: Numerator corrections of propagator $\pi_x\pi_x$

$Fig.\ (7a) \sim O\left(\omega^{\frac{3}{2}}\right)$ $\quad$ $Fig.\ (7b) \sim \#\Lambda$ term $\quad$ $Fig.\ (7c) \sim O\left(\omega^{\frac{3}{2}}\right)$

$Fig.\ (8a) \sim O\left(\omega^{\frac{5}{2}}\right)$ $\quad$ $Fig\ (8b, 8c,) \sim O\left(\omega^{\frac{3}{2}}\right)$ $\quad$ $Fig.\ (8d) \sim \#\Lambda$ term $\quad$ $Fig.\ (8e) \sim O\left(\omega^{\frac{3}{2}}\right)$

$$\lambda_1 \sim \frac{\omega^{\frac{1}{2}}}{k}$$

$$\lambda_2 \sim \epsilon_{\varepsilon n}\frac{\omega^{\frac{1}{2}}}{k}$$

$$\lambda_3 \sim \frac{\omega^{\frac{1}{4}}}{k^{\frac{1}{2}}}$$

work in progress

Summary and Outlooks

- Applying the lattice equation of state
- Considering the gradient terms such as $n\nabla^2 n$ in the free energy to describe the effect of phase interface in the liquid-gas phase transition
- Extending such strategy in the non-equilibrium steady state system within a stable distribution
- Investigating the critical behaviors to the second order hydro fluctuations

Thank You for Your Attention!