# Modification to fluctuation-dissipation relation and fluctuation theorem

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KM, Annals Phys. **411**, 167969 (2019) [arXiv:1904.11217]

New Development of Hydrodynamics and its applications in Heavy-ion Collisions,
Fudan University, Shanghai, Oct 31, 2019

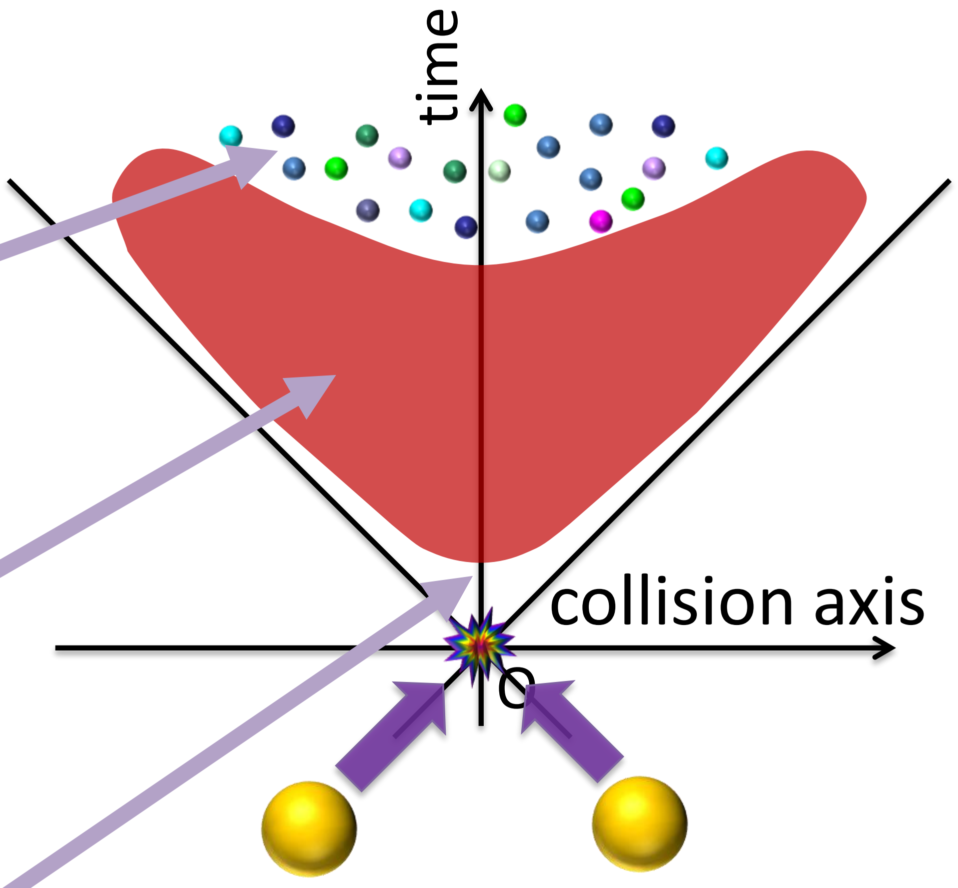
# Dynamical models for heavy-ion collisions

Final observables

Hadronic cascades

Hydrodynamics

Initialization model



# Fluctuations/Correlations

Final observables

Hadronic cascades

Hydrodynamics

Initialization model

etc.

⋮

Non-flow contributions

Jet-induced medium excitations

<u>Hydrodynamic fluctuations</u>: thermal fluctuations of hydrodynamics

Initial state fluctuations

# Fluctuating hydrodynamics

= viscous hydrodynamics with noise terms

**Stochastic partial differential equations (SPDE)**

**Conservation law** $\partial_\mu T^{\mu\nu} = 0.$

hydrodynamic fluctuations
→ noise terms

**Constitutive eqs.**

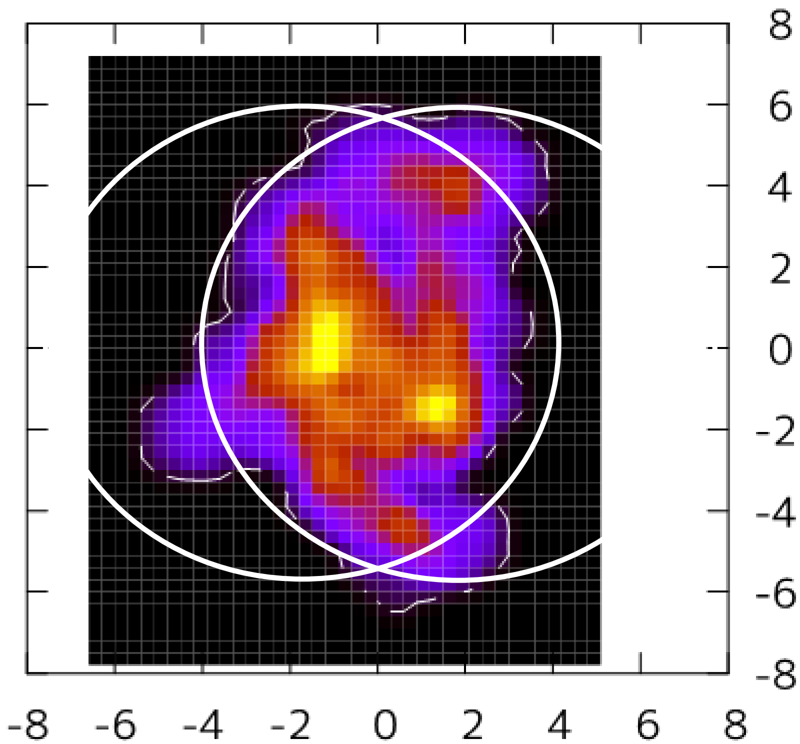
$$\pi^{\mu\nu} + \tau_\pi \Delta^{\mu\nu}{}_{\alpha\beta} \mathrm{D}\pi^{\alpha\beta} = 2\eta\partial^{\langle\mu}u^{\rangle\nu} + \cdots + \xi_\pi^{\mu\nu},$$
$$[1 + \tau_\Pi \mathrm{D}]\Pi = -\zeta\partial_\mu u^\mu + \cdots + \xi_\Pi.$$

**Fluctuation-dissipation relation** (FDR)

$$\langle \xi^{\mu\nu}(x)\xi^{\alpha\beta}(x')\rangle = 4T\eta\Delta^{\mu\nu\alpha\beta}\delta^{(4)}(x-x'),$$
$$\langle \xi(x)\xi(x')\rangle = 2T\zeta\delta^{(4)}(x-x'),$$

# Inhomogeneous and non-static matter

Matter created in nuclear collisions
is <u>not</u> ***static*** and ***homogeneous***




**Au+Au 200GeV MC-KLN**
**(*x*-*y* plane)**

Usual FDR relies on
the linear-response
of ***global equilibrium***
to small perturbations

How is FDR modified in
***inhomogeneous*** and
***non-static*** matter?

# Procedure (1/4)

**Step 1** **Define (linear-response) constitutive equation (CE)**
in inhomogeneous and non-static background

**"Differential form of CE"**

$$L_x \underline{\Gamma(x)} = M_x \underline{\kappa(x)F(x)} + \underline{\xi(x)}$$

$\Gamma$: dissipative current

$\pi^{\mu\nu}$, $\Pi$, $\nu^{\mu}$

$\kappa$: 1st order transport coefficient

$F$: thermodynamic force

$\xi$: noise term

$L_x$, $M_x$: linear operators

**e.g. Bulk pressure**

$$L = 1 + \tau_\Pi \mathrm{D},$$
$$M = 1$$

$$\Rightarrow \quad \tau_\Pi D\Pi + \Pi = -\zeta\theta$$

# Procedure (2/4)

**Step 2** **Solve CE to obtain the integral form**

**“Integral form of CE”**

$$\Gamma(x) = \int dy\, G(x,y)\kappa(y)F(y) + \delta\Gamma(x)$$

Obtain explicit form of *memory function*

$$G(x,y) = L_x^{-1} M_x\, \delta(x-y) = \dots$$

$\delta\Gamma$: integral form noise

$$\xi(x) = L_x \delta\Gamma(x)$$

# Procedure (3/4)

**Step 3** **Use FDR based on Zubarev's NESO (Non-equilibrium statistical operator)**

See, e.g., D. N. Zubarev, Nonequilibrium Statistical Thermodynamics (Plenum, New York, 1974), A. Hosoya, M.-a. Sakagami, and M. Takao, Annals Phys. 154, 229 (1984).

$$\langle \delta\Gamma(x)\delta\Gamma(y)\rangle \Theta(x^0 - y^0) = G(x,y)T(y)\kappa(y)$$

**Autocorrelation of integral form noise δΓ**

$$\langle \delta\Gamma(x)\delta\Gamma(y)\rangle = \underbrace{G(x,y)}_{\text{From Step 2}}T(y)\kappa(y) + T(y)\kappa(y)\underbrace{G(y,x)}_{\text{From Step 2}}$$

# Procedure (4/4)

**Step 4** **Obtain noise autocorrelation in original CE**

**Autocorrelation for differential form noise δΓ**

$$\langle \xi(x)\xi(y) \rangle = L_x L_y \underbrace{\langle \delta\Gamma(x)\delta\Gamma(y) \rangle}_{\text{From Step 3}} = \ldots$$

# E.g. Simplified Israel-Stewart

**Step 1** Input CE

$$(1+\tau_\Pi\,\mathrm{D})\Pi = -\zeta\theta + \xi_\Pi,$$ **Bulk pressure**

$$(1+\tau_\pi\mathcal{D})\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \xi_\pi^{\mu\nu},$$ **Shear stress**

$$(\delta_{ij}+\tau_{ij}\mathcal{D})\nu_j^\mu = \kappa_{ij}T\nabla^\mu\frac{\mu_j}{T} + \xi_i^\mu.$$ **Diffusion current**

Defs: substantial time derivatives

$$\mathrm{D} = u^\mu\partial_\mu,$$

$$\mathcal{D}\pi^{\mu\nu} = \Delta^{\mu\nu}{}_{\alpha\beta}\,\mathrm{D}\,\pi^{\alpha\beta},$$

$$\mathcal{D}\nu_i^\mu = \Delta^\mu{}_\alpha\,\mathrm{D}\,\nu_i^\alpha.$$

# E.g. Simplified Israel-Stewart

## Step 2 Solve $G(x,y)$

$$G_\Pi(x,x') = \exp\left(-\int_{\tau(x')}^{\tau(x)} \frac{d\tau_2}{\tau_\Pi(\tau_2,\boldsymbol{\sigma}(x))}\right)\frac{1}{\tau_\Pi(x')}\theta^{(4)}(\sigma(x),\sigma(x')),$$

$$G^{\mu\nu}{}_{\alpha\beta}(x,x') = \exp\left(-\int_{\tau(x')}^{\tau(x)} \frac{d\tau_2}{\tau_\pi(\tau_2,\boldsymbol{\sigma}(x))}\right)\frac{1}{\tau_\pi(x')}\underline{\Delta(\tau(x);\tau(x'),\boldsymbol{\sigma}(x))^{\mu\nu}{}_{\alpha\beta}}\,\theta^{(4)}(\sigma(x),\sigma(x')),$$

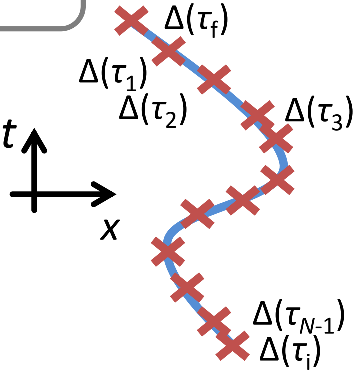
$$G_{ik}{}^{\mu}{}_{\alpha}(x,x') = \sum_{j=1}^{n}\left[\mathrm{T}\exp\left(-\int_{\tau(x')}^{\tau(x)} d\tau_2\,\tau_{ij}^{-1}(\tau_2,\boldsymbol{\sigma}(x))\right)\right]_{ij}\tau_{jk}^{-1}(x')\underline{\Delta(\tau(x);\tau(x'),\boldsymbol{\sigma}(x))^{\mu}{}_{\alpha}}\,\theta^{(4)}(\sigma(x),\sigma(x')).$$

Complicated tensor structure due to projection in CE

$$\mathcal{D}\pi^{\mu\nu} = \Delta^{\mu\nu}{}_{\alpha\beta}\,\mathrm{D}\,\pi^{\alpha\beta},$$
$$\mathcal{D}\nu_i^{\mu} = \Delta^{\mu}{}_{\alpha}\,\mathrm{D}\,\nu_i^{\alpha}.$$

$$\Delta(\tau_\mathrm{f};\tau_\mathrm{i})^{\mu}{}_{\alpha} := \lim_{N\to\infty}\Delta(\tau_\mathrm{f})^{\mu}{}_{\alpha_0}\left[\prod_{k=0}^{N-1}\Delta(\tau_\mathrm{f}+\tfrac{\tau_\mathrm{i}-\tau_\mathrm{f}}{N}k)^{\alpha_k}{}_{\alpha_{k+1}}\right]\Delta(\tau_\mathrm{i})^{\alpha_N}{}_{\alpha},$$

$$\Delta(\tau_\mathrm{f};\tau_\mathrm{i})^{\mu\nu}{}_{\alpha\beta} := \lim_{N\to\infty}\Delta(\tau_\mathrm{f})^{\mu\nu}{}_{\alpha_0\beta_0}\left[\prod_{k=0}^{N-1}\Delta(\tau_\mathrm{f}+\tfrac{\tau_\mathrm{i}-\tau_\mathrm{f}}{N}k)^{\alpha_k\beta_k}{}_{\alpha_{k+1}\beta_{k+1}}\right]\Delta(\tau_\mathrm{i})^{\alpha_N\beta_N}{}_{\alpha\beta}.$$



# E.g. Simplified Israel-Stewart

## Step 4 Resulting FDR for differential form CE

$$
\begin{aligned}
\langle \xi_\Pi(x)\xi_\Pi(x')\rangle &= \left(2 + \boxed{\tau_\Pi \mathrm{D}\ln\frac{T\zeta}{\tau_\Pi} - \tau_\Pi\theta}\right) T\zeta\delta^{(4)}(x-x'),\\
\langle \xi_\pi^{\mu\nu}(x)\xi_\pi^{\alpha\beta}(x')\rangle &= 2\left[\left(2 + \boxed{\tau_\pi \mathrm{D}\ln\frac{T\eta}{\tau_\pi} - \tau_\pi\theta}\right)\Delta^{\mu\nu\alpha\beta} + \boxed{\tau_\pi\mathcal{D}\Delta^{\mu\nu\alpha\beta}}\right] T\eta\delta^{(4)}(x-x'),\\
\langle \xi_i^{\mu}(x)\xi_j^{\alpha}(x')\rangle &= -2T\kappa_{ij}\Delta^{\mu\alpha}\delta^{(4)}(x-x')\\
&\quad - \Delta^{\mu\alpha}[K^{\mathrm{A}}_{ij}(x)\mathcal{D} - K^{\mathrm{A}}_{ij}(x')\mathcal{D}']\delta^{(4)}(x-x')\\
&\quad + \sum_{k=1}^{n}\left\{\boxed{-\Delta^{\mu\alpha}\left[\tau_{ik}\mathrm{D}T\kappa_{kj} - (\mathrm{D}\tau_{ik})T\kappa_{kj} - \tau_{ik}\theta T\kappa_{kj}\right]^{\mathrm{S}} - K^{\mathrm{S}}_{ij}\mathcal{D}\Delta^{\mu\alpha}}\right\}\delta^{(4)}(x-x'),
\end{aligned}
$$

where $K^{\mathrm{S/A}}_{ij}(x) = \sum_{k=1}^{n} T(x)(\tau_{ik}(x)\kappa_{kj}(x) \pm \tau_{jk}(x)\kappa_{ki}(x))/2$, and $[\circ_{ij}]^{\mathrm{S}} = (\circ_{ij} + \circ_{ji})/2$.

**Complicated… but essential structure is:**

$$\langle \xi(x)\xi(x')\rangle = \left(2 + \boxed{\tau_R \mathrm{D}\ln\frac{T\kappa}{\tau_R} - \tau_R\theta}\right) T\kappa\delta^{(4)}(x-x').$$

# E.g. Simplified Israel-Stewart

**Step 4** Resulting FDR for differential form CE

$$\langle \xi(x)\xi(x')\rangle = \left(2 + \boxed{\tau_R \mathrm{D} \ln \frac{T\kappa}{\tau_R} - \tau_R\theta}\right) T\kappa\delta^{(4)}(x-x').$$

✓ **Modification to FDR ∝ Relaxation time $\tau_R$ and expansion $\theta$**

Important in HIC
with the short-scale ($\sim\tau_R$) dynamics
of *rapidly expanding* system

# Fluctuation theorem (FT)

D. J. Evans, E. G. D. Cohen, G. P. Morriss, Phys. Rev. Lett. **71**, 2401–2404 (1993)

Relations for probability density **Pr(δ*S*)** of entropy production **δ*S***

known in non-equilibrium statistical mechanics:

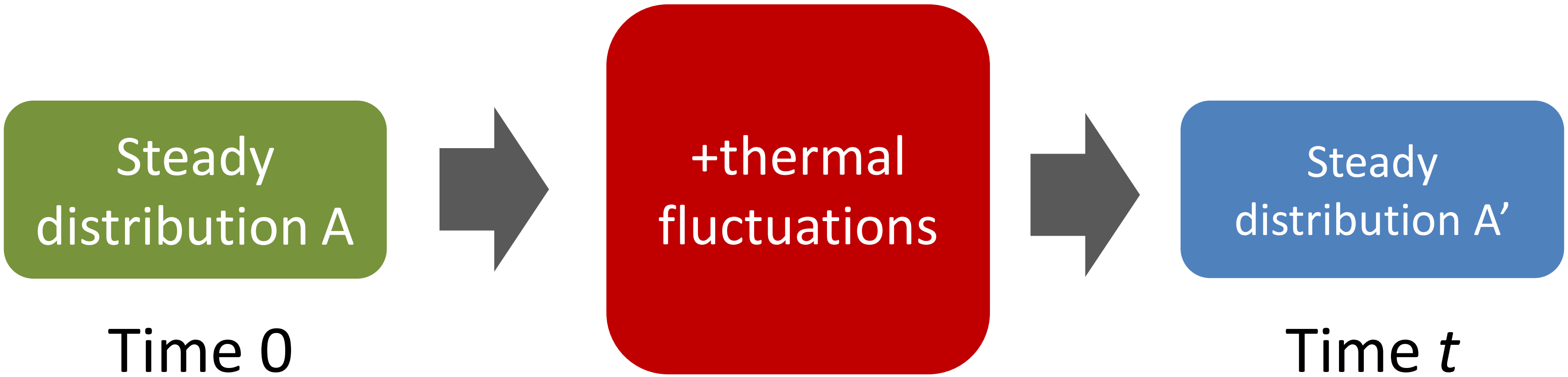
$$\ln \frac{\Pr(\delta S = \alpha)}{\Pr^{\dagger}(\delta S^{\dagger} = -\alpha)} = \alpha$$

Note: definitions of $\delta S$, $\delta S^{\dagger}$, $\Pr(\delta S)$, $\Pr^{\dagger}(\delta S^{\dagger})$ *depends on systems and processes*.

→ many variations of FT

# Fluctuation theorem (FT)

## (Example) Steady-state FT (**SSFT**)




$$\ln \frac{\Pr(\bar{\sigma} = \alpha)}{\Pr(\bar{\sigma} = -\alpha)} = \alpha t + O(\tau_R / t)$$

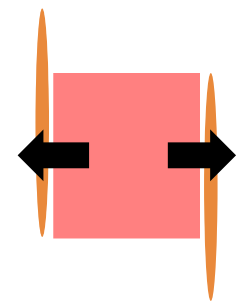
- $\sigma = \delta S/t$: entropy production rate per unit time
- $\tau_R$: relaxation time scale of the system

# FT in Bjorken flow

Let's check the FT
in 0+1D Bjorken flow with noise terms

***Steady-state FT***

$$\ln \frac{\Pr(\bar{\sigma} = \alpha)}{\Pr(\bar{\sigma} = -\alpha)} = \alpha t + O(\tau_R / t)$$



$\sigma$: entropy production rate $\quad \bar{\sigma} = \dfrac{s(\tau) - s(\tau_0)}{\tau - \tau_0}$

*When the distribution is Gaussian*

$$R := \frac{2\langle \bar{\sigma} \rangle}{a^2 \cdot (\tau - \tau_0)} \to 1.$$

Calculate this quantity for FT

T. Hirano, R. Kurita, KM, Nucl. Phys. A 984 (2019) 44-67

$a^2$: variance of $\sigma$

# FT in Bjorken flow: Hydro eqs

## (0+1)-dim Bjorken flow
## 2nd order fluctuating hydro

**Conservation**

$\tau$: proper time

$$\frac{\mathrm{d}e}{\mathrm{d}\tau} = -\frac{e+p}{\tau}\left(1 - \frac{\pi - \Pi}{sT}\right)$$

**Constitutive eqs. (simplified IS)**

$\pi = \pi^{00} - \pi^{33}$

$$\left(\tau_\pi \frac{\mathrm{d}}{\mathrm{d}\tau} + 1\right)\pi = \frac{4\eta}{3\tau} + \xi_\pi,$$

$$\left(\tau_\Pi \frac{\mathrm{d}}{\mathrm{d}\tau} + 1\right)\Pi = -\frac{\zeta}{\tau} + \xi_\Pi.$$

Hydrodynamic fluctuations

Gaussian noise

# FT in Bjorken flow: FDR

## (0+1)-dim Bjorken flow
## $2^{nd}$ order fluctuating hydro

**Case 1**: **FDR Without modification**

$$\langle \xi_\pi(\tau)\xi_\pi(\tau')\rangle = \frac{8T\eta\delta(\tau-\tau')}{3\tau\Delta\eta_s\Delta x\Delta y},$$

$$\langle \xi_\Pi(\tau)\xi_\Pi(\tau')\rangle = \frac{2T\zeta\delta(\tau-\tau')}{\tau\Delta\eta_s\Delta x\Delta y}.$$

Cell

$$V(\tau) = \tau\Delta\eta_s\Delta x\Delta y$$

**Case 2**: **FDR With modification**

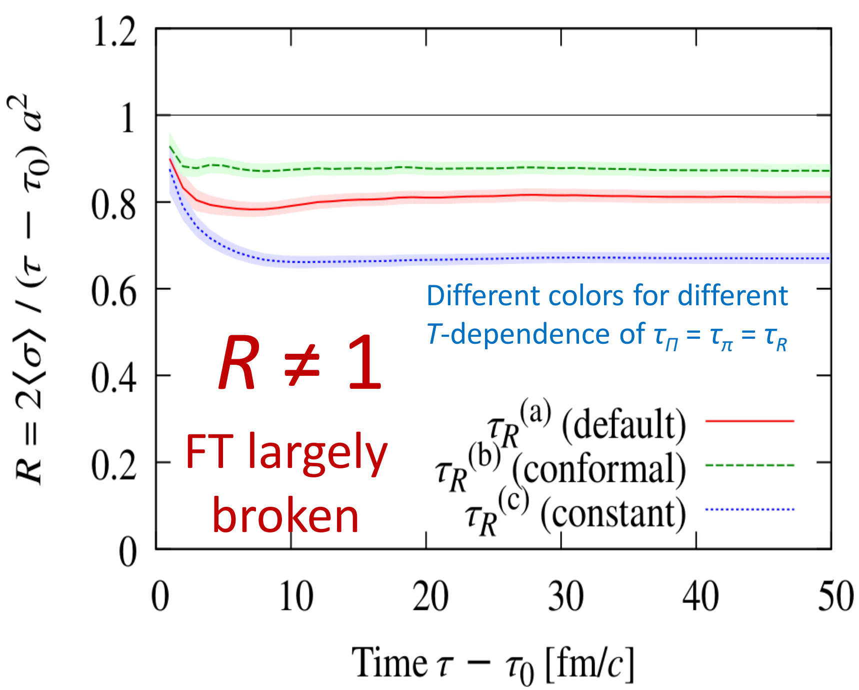
$$\langle \xi_\pi(\tau)\xi_\pi(\tau')\rangle = \frac{8T\eta\delta(\tau-\tau')}{3\tau\Delta\eta_s\Delta x\Delta y}\left(1-\frac{T\eta}{2}\partial_\mu\frac{\tau_\pi u^\mu}{T\eta}\right),$$

$$\langle \xi_\Pi(\tau)\xi_\Pi(\tau')\rangle = \frac{2T\zeta\delta(\tau-\tau')}{\tau\Delta\eta_s\Delta x\Delta y}\left(1-\frac{T\zeta}{2}\partial_\mu\frac{\tau_\Pi u^\mu}{T\zeta}\right).$$

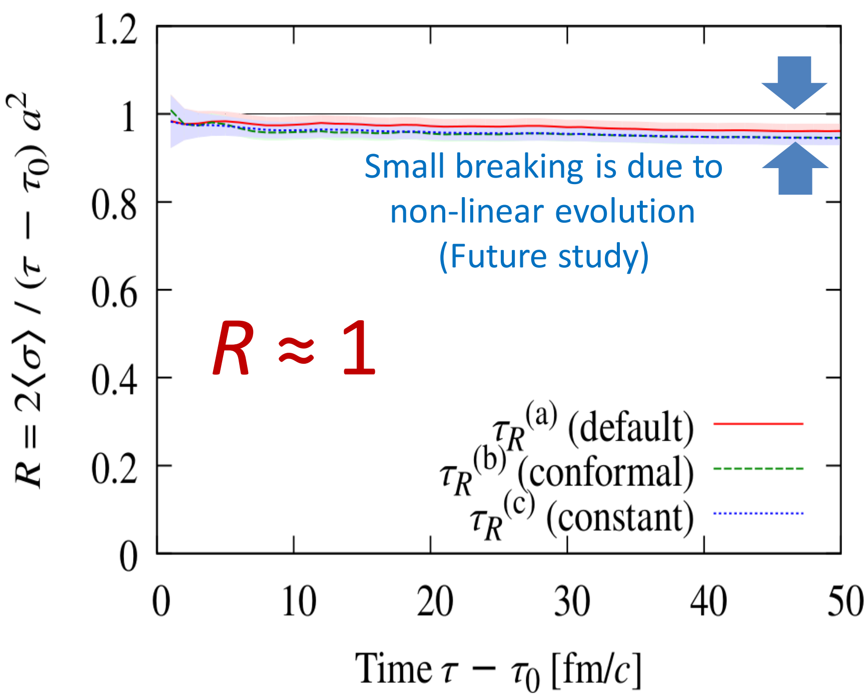
Modification

# FT in Bjorken flow: Result

**Case 1: Without FDR modification**




**Case 2: With FDR modification**




Without FDR modification, FT is violated.

✓ **FDR modification is needed to satisfy FT**

# SUMMARY

# Summary

- ***Fluctuation-dissipation relation (FDR)*** *for second-order hydro* has ***modifications proportional to*** $\boldsymbol{\tau_R}$ in inhomogeneous and non-static systems

$$\langle \xi(x)\xi(x')\rangle = \left(2 + \boxed{\tau_R \mathrm{D} \ln \frac{T\kappa}{\tau_R} - \tau_R \theta}\right) T\kappa \delta^{(4)}(x - x').$$

- ***Steady-state Fluctuation theorem*** is ***broken without FDR modification***.
- FDR modifications should be incorporated into dynamical models with finite relaxation times

# BACKUP

# Simplified Israel-Stewart eq.

## Solving the simplified Israel-Stewart equation

$$\Gamma = \kappa F - \tau_R \mathcal{D}\Gamma$$

$$\left\{\begin{aligned}
\Gamma &= (\Pi, \pi^{\mu\nu}, \nu_i^{\mu})^{\mathrm{T}}, && \text{dissipative currents} \\
\kappa &= \mathrm{diag}(\zeta, 2\eta\Delta^{\mu\nu\alpha\beta}, -\kappa_{ij}\Delta^{\mu\alpha}), && \text{Onsager coefficients} \\
F &= \left(-\theta, \partial_{\langle\alpha}u_{\beta\rangle}, -T\nabla_{\alpha}\frac{\mu_i}{T}\right)^{\mathrm{T}}, && \text{thermodynamic force} \\
\mathcal{D} &= \Delta \mathrm{D}, && \text{time derivative} \\
\Delta &= \mathrm{diag}(1, \Delta^{\mu\nu}{}_{\alpha\beta}, \Delta^{\mu}{}_{\alpha}), && \text{projection} \\
\tau_R &= \mathrm{diag}(\tau_{\Pi}, \tau_{\pi}, \tau_{ij}), && \text{relaxation time}
\end{aligned}\right.$$

spacetime dependent projections

# Projection and transversality

## Transverse condition for dissipative currents

$$u_\mu \pi^{\mu\nu} = u_\mu \nu_i^\mu = 0$$

## Projection is needed to maintain the *transversality*

$$\tau_\pi \underline{\Delta^{\mu\nu}{}_{\alpha\beta}} \mathrm{D}\pi^{\alpha\beta} = -\pi^{\mu\nu} + 2\eta \partial^{\langle\mu} u^{\nu\rangle}$$

**without projection…** $\tau_\pi \mathrm{D}\pi^{\mu\nu} = -\pi^{\mu\nu} + 2\eta \partial^{\langle\mu} u^{\nu\rangle}$

$$\mathrm{D}(u_\mu \pi^{\mu\nu}) = \pi^{\mu\nu}\mathrm{D}u_\mu + u_\mu \tau_\pi^{-1}(2\eta\partial^{\langle\mu} u^{\nu\rangle} - \pi^{\mu\nu}) = \pi^{\mu\nu}\mathrm{D}u_\nu$$

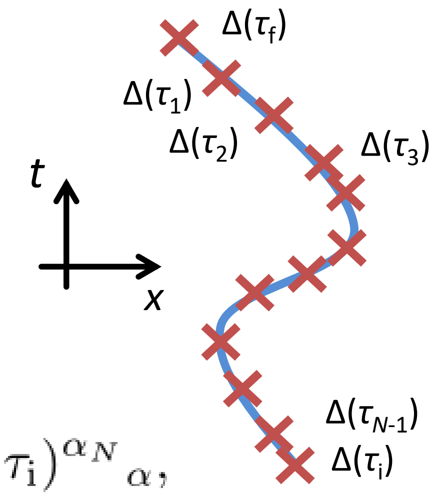
transversality would break if background changes (**$Du_\nu \neq 0$**)

# New projection

## Projections along the pathline

pathline: the world line of a fluid particle

$\tau$: the proper time of the fluid particle




$$\Delta(\tau_\mathrm{f};\tau_\mathrm{i})^\mu{}_\alpha := \lim_{N\to\infty} \Delta(\tau_\mathrm{f})^\mu{}_{\alpha_0} \left[ \prod_{k=0}^{N-1} \Delta(\tau_\mathrm{f} + \tfrac{\tau_\mathrm{i}-\tau_\mathrm{f}}{N} k)^{\alpha_k}{}_{\alpha_{k+1}} \right] \Delta(\tau_\mathrm{i})^{\alpha_N}{}_\alpha ,$$

$$\Delta(\tau_\mathrm{f};\tau_\mathrm{i})^{\mu\nu}{}_{\alpha\beta} := \lim_{N\to\infty} \Delta(\tau_\mathrm{f})^{\mu\nu}{}_{\alpha_0\beta_0} \left[ \prod_{k=0}^{N-1} \Delta(\tau_\mathrm{f} + \tfrac{\tau_\mathrm{i}-\tau_\mathrm{f}}{N} k)^{\alpha_k\beta_k}{}_{\alpha_{k+1}\beta_{k+1}} \right] \Delta(\tau_\mathrm{i})^{\alpha_N\beta_N}{}_{\alpha\beta} .$$

## Properties

$$\mathrm{D_f}\Delta(\tau_\mathrm{f};\tau_\mathrm{i})^\mu{}_\alpha = [\mathrm{D_f}\Delta(\tau_\mathrm{f})^\mu{}_\kappa]\Delta(\tau_\mathrm{f};\tau_\mathrm{i})^\kappa{}_\alpha ,$$

$$\mathrm{D_f}\Delta(\tau_\mathrm{f};\tau_\mathrm{i})^{\mu\nu}{}_{\alpha\beta} = [\mathrm{D_f}\Delta(\tau_\mathrm{f})^{\mu\nu}{}_{\kappa\lambda}]\Delta(\tau_\mathrm{f};\tau_\mathrm{i})^{\kappa\lambda}{}_{\alpha\beta}$$ , etc.

# FT in high-energy nuclear collisions

**<u>FT: generalization of FDR near the equilibrium</u>**

→ Relations to FDR in fluctuating hydrodynamics?

→ Entropy distribution through multiplicity?

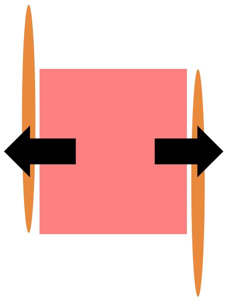
Qualitative understanding
under idealized assumptions

**<u>Simplification</u>**

① **Bjorken flow**: (0+1)-dim evolution

② **Linear fluctuations**: no non-linear fluctuations

③ **Navier-Stokes limit**: negligible $\tau_R$



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# Hydrodynamic equations

## (0+1)-dim Bjorken flow (assumption ①)

## 2nd order fluctuating hydro

**Conservation**

$\tau$: proper time

$$\frac{\mathrm{d}e}{\mathrm{d}\tau} = -\frac{e+p}{\tau}\left(1 - \frac{\pi - \Pi}{sT}\right)$$

**Constitutive eqs.**

$$\pi = \pi^{00} - \pi^{33}$$

$$\left(\tau_\pi \frac{\mathrm{d}}{\mathrm{d}\tau} + 1\right)\pi = \frac{4\eta}{3\tau} + \underline{\xi_\pi},$$

$$\left(\tau_\Pi \frac{\mathrm{d}}{\mathrm{d}\tau} + 1\right)\Pi = -\frac{\zeta}{\tau} + \underline{\xi_\Pi}.$$

Hydrodynamic fluctuations

Gaussian noise

**FDR**

Cell

$$V(\tau) = \tau \Delta\eta_s \Delta x \Delta y$$

$$\langle \xi_\pi(\tau)\xi_\pi(\tau')\rangle = \frac{8\eta T}{3\tau\Delta\eta_s\Delta x\Delta y}\delta(\tau - \tau'),$$

$$\langle \xi_\Pi(\tau)\xi_\Pi(\tau')\rangle = \frac{2\zeta T}{\tau\Delta\eta_s\Delta x\Delta y}\delta(\tau - \tau').$$



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# Entropy production rate

**Step 1. Def. Entropy production in a cell**

$$\bar{\sigma} := \frac{s(\tau)V(\tau) - s(\tau_{\mathrm{i}})V(\tau_{\mathrm{i}})}{\tau - \tau_{\mathrm{i}}}.$$

*s*: Equilibrium entropy
$V(\tau)$: volume of the cell
$\tau_{\mathrm{i}}$: initial time

**Step 2. Result of time evolution**

$$\bar{\sigma} = \frac{1}{\tau - \tau_{\mathrm{i}}} \int_{\tau_{\mathrm{i}}}^{\tau} \mathrm{d}\tau' \frac{\pi(\tau') - \Pi(\tau')}{T(\tau')} \Delta\eta_s \Delta x \Delta y.$$

where

$$\pi(\tau) = \int_{\tau_{\mathrm{i}}}^{\tau} \mathrm{d}\tau' G_{\pi}(\tau, \tau') \frac{4\eta}{3\tau'} + \delta\pi(\tau), \qquad \delta\pi(\tau) = \int_{\tau_{\mathrm{i}}}^{\tau} \mathrm{d}\tau' G_{\pi}(\tau, \tau') \xi_{\pi}(\tau'),$$

$$\Pi(\tau) = -\int_{\tau_{\mathrm{i}}}^{\tau} \mathrm{d}\tau' G_{\Pi}(\tau, \tau') \frac{\zeta}{\tau'} + \delta\Pi(\tau), \qquad \delta\Pi(\tau) = \int_{\tau_{\mathrm{i}}}^{\tau} \mathrm{d}\tau' G_{\Pi}(\tau, \tau') \xi_{\Pi}(\tau').$$

$$G_{\pi/\Pi}(\tau_2, \tau_1) := \exp\left( -\int_{\tau_1}^{\tau_2} \frac{\mathrm{d}\tau}{\tau_{\pi/\Pi}(\tau)} \right) \frac{1}{\tau_{\pi/\Pi}(\tau_1)}$$


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# Entropy production rate

## Step 3. Distribution of entropy production

Linear fluctuations (assumption ②) → Gaussian distributions

Navier-Stokes limit (assumption ③) → Simplified expr.

**average** $$\langle\bar{\sigma}\rangle = \frac{\Delta\eta_s\Delta x\Delta y}{\tau-\tau_\mathrm{i}}\boxed{\int_{\tau_\mathrm{i}}^{\tau}\frac{\mathrm{d}\tau'}{T_0(\tau')}\left(\frac{4\eta}{3\tau'}+\frac{\zeta}{\tau'}\right)},$$

**variance** $$\begin{aligned} a^2 &= \langle\bar{\sigma}^2\rangle - \langle\bar{\sigma}\rangle^2 \\ &= \frac{2\Delta\eta_s\Delta x\Delta y}{(\tau-\tau_\mathrm{i})^2}\boxed{\int_{\tau_\mathrm{i}}^{\tau}\frac{\mathrm{d}\tau'}{T_0(\tau')}\left(\frac{4\eta}{3\tau'}+\frac{\zeta}{\tau'}\right)} \end{aligned}$$

→ $$\frac{2\langle\bar{\sigma}\rangle}{a^2} = \tau - \tau_\mathrm{i} \quad \Leftrightarrow \quad \ln\frac{\mathrm{Pr}(\bar{\sigma}=\alpha)}{\mathrm{Pr}(\bar{\sigma}=-\alpha)} = \alpha t + O(\tau_R/t)$$

✓ Expression equivalent to SSFT in expanding system



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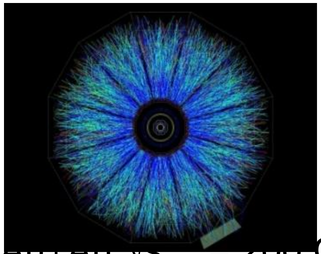
# Multiplicity fluctuations

## In high-energy nuclear collisions?



Au+Au $\sqrt{s_{NN}}$ = 200 GeV
STAR Collaboration (RHIC)

### Upper bound in entropy fluctuations

$$\frac{\Delta S(\tau)}{\langle S(\tau)\rangle} = \frac{a(\tau-\tau_i)}{S_i + \langle\bar{\sigma}\rangle(\tau-\tau_i)}$$

*S*: entropy

$S_i$: initial entropy

$$= \frac{\sqrt{2\langle\bar{\sigma}\rangle(\tau-\tau_i)}}{S_i + \langle\bar{\sigma}\rangle(\tau-\tau_i)}$$

∵ SSFT $\frac{2\langle\bar{\sigma}\rangle}{a^2} = \tau - \tau_{\mathrm{i}}$

$$\leq \frac{1}{\sqrt{2S_{\mathrm{i}}}}$$

∵ inequality $\frac{\sqrt{2y}}{x+y} \leq \frac{1}{\sqrt{2x}}$

### → Upper bound of multiplicity fluctuations

$$\frac{(\Delta_{\mathrm{ev}} N)^2 - \underbrace{\langle N\rangle_{\mathrm{ev}}}_{\text{Poisson statistics}}}{\langle N\rangle_{\mathrm{ev}}^2} \leq \underbrace{\frac{(\Delta_{\mathrm{ev}} S_{\mathrm{tot,i}})^2}{\langle S_{\mathrm{tot,i}}\rangle_{\mathrm{ev}}^2}}_{\text{Initial state fluctuations}} + \underbrace{\frac{1}{2\langle S_{\mathrm{tot,i}}\rangle_{\mathrm{ev}}}}_{\text{SSFT upper bound}}$$

**LHS**: observables
**RHS**: initial state
→constraining initial state independently of intermediate dynamics?

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