Formulation of relativistic spin hydrodynamics based on entropy-current analysis

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with K. Hattori (YITP), M. Hongo (Illinois), X.-G. Huang (Fudan), M. Matsuo (UCAS)

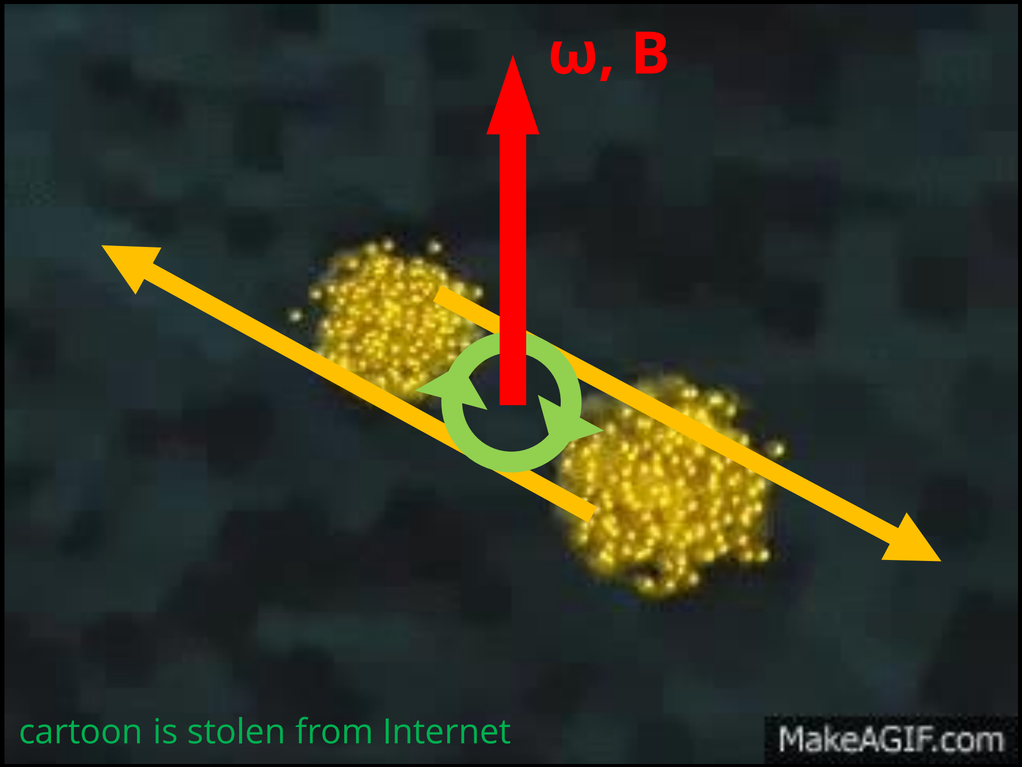
Hattori, Hongo, Huang, Matsuo, HT, Phys. Lett. B795, 100 (2019) [arXiv:1901.06615]

Ultra-relativistic heavy ion collisions

Aim: study quark-gluon plasma (QGP)

Lesson: QGP behaves like a perfect liquid
and **hydrodynamics works so well**

Huge ω and B



Question: QGP under huge ω and/or B ?

Expectation: QGP is polarized

cf. talk by Becattini, Xia, ...

✓ **Magnetic field B effect**

Zeeman splitting (Landau quantization)

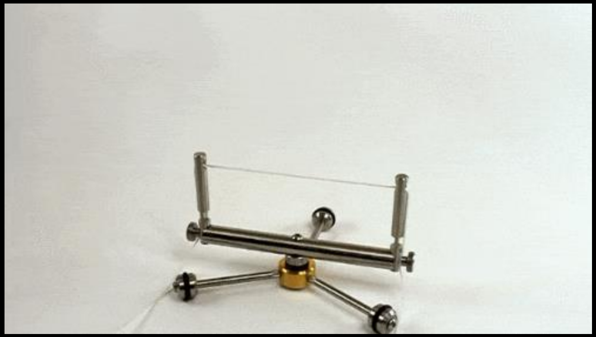
$$E \to E - s \cdot qB$$

➡ **charge dependent spin polarization**

✓ **Rotation ω effect**

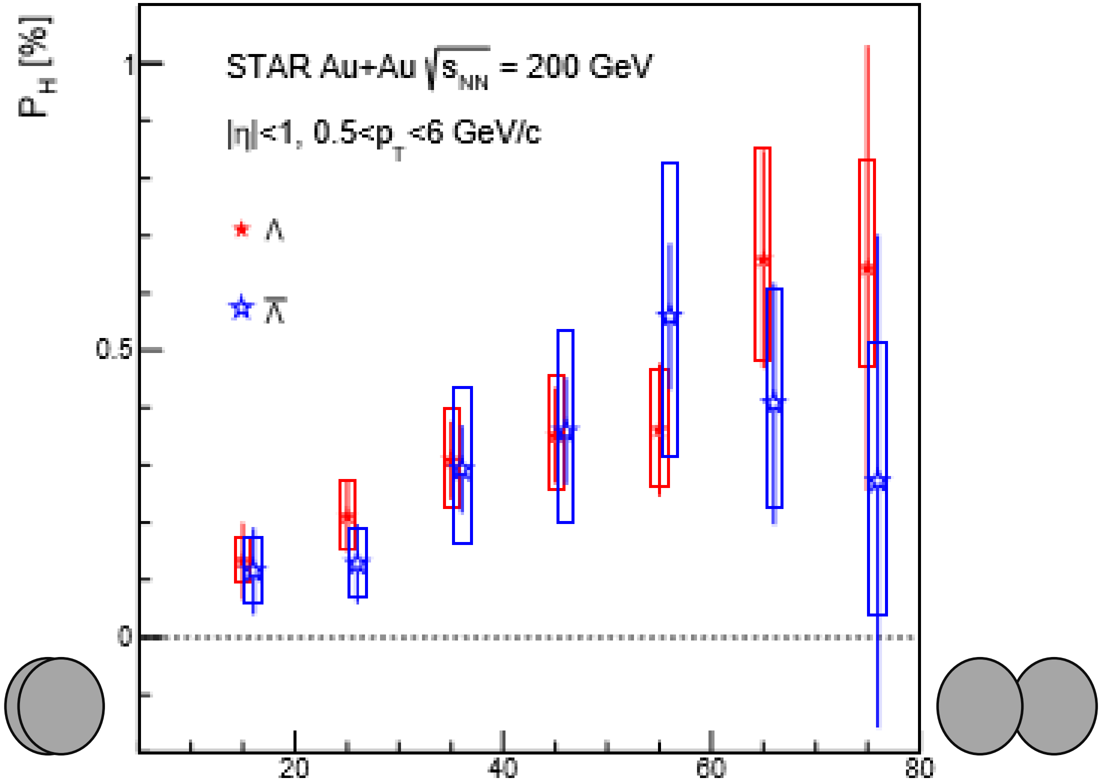
Barnett effect

$$E \to E - s \cdot \omega$$



➡ **charge independent spin polarization**

Experimental fact ➡ Observed



[STAR, PRC 98, 014910 (2018)]

FIG. 5. Λ ($\bar{\Lambda}$) polarization as a function of the collision centrality in Au+Au collisions at $\sqrt{s_{NN}}$ = 200 GeV. Open boxes and vertical lines show systematic and statistical uncertainties. The data points for $\bar{\Lambda}$ are slightly shifted for visibility.

How about theory?

Hydrodynamics for spin polarized QGP?

<u>Far from complete</u>

Hydrodynamics for spin polarized QGP

✓ **"Hydro simulations" exist, but…**

- usual hydro (i.e., hydro **w/o** spin) is solved
- thermal vorticity $\varpi^{\mu\nu} \equiv \partial^\mu(u^\nu/T) - \partial^\nu(u^\mu/T)$ is converted into spin via Cooper-Frye formula (???)

✓ **Formulation of relativistic hydrodynamics with spin is still under construction**

cf. talk by Wojciech

Current status of formulation of spin hydro

✓ **Non-relativistic case**

e.g. Eringen (1998); Lukaszewicz (1999)

Already established (e.g. micropolar fluid)

- applied to pheno. and is successful

e.g. spintronics: Takahashi et al. (2015)

- **spin must be dissipative** because of mutual conversion b/w spin and orbital angular momentum

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✓ **Relativistic case**

Some preceding works do exist, but

- only for "ideal" fluid (**no dissipative corrections**)
- some claim **spin should be conserved**

Purpose of this talk

- ✓ Formulate **relativistic spin hydro with 1st order dissipative corrections** for the first time
- ✓ Clarify spin must be **dissipative**

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Outline

1. ~~Introduction~~
2. Formulation based on entropy-current analysis
3. Linear mode analysis
4. Summary

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e.g. kinetic theory; QFT; Lagrangian; fluid/gravity; projection op. …

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  (see talk by Hattori)

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T^{\mu\nu} = {} & f_1(\beta) g^{\mu\nu} + f_2(\beta) u^\mu u^\nu \\
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& + f_6(\beta) g^{\mu\nu} \partial^\rho u_\rho + f_7(\beta) u^\mu u^\nu \partial^\rho u_\rho + f_8(\beta) u^\mu \partial_\mu u^\nu + \cdots + O(\partial^2)
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  1. symmetry
  2. power counting ➡ **gradient expansion**
  3. other physical requirements ➡ **thermodynamics** *(see next slide)*

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✓ **Hydrodynamic eq. = conservation law + constitutive relation**

Formulation of hydro w/o spin (2/3)

✓ **Constraints by thermodynamics**

Formulation of hydro w/o spin (2/3)

✓ Constraints by thermodynamics

Expand $T^{\mu\nu}$ i.t.o derivatives

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)} + O(\partial^2) \quad \text{where} \quad T^{\mu\nu}_{(0)} = eu^\mu u^\nu + p(g^{\mu\nu} + u^\mu u^\nu)$$

because $T^{\mu\nu} \xrightarrow[\text{static eq.}]{} T^{\mu\nu}_{(0)} = (e, p, p, p)$

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1st law of thermodynamics says

$$ds = \beta de, \ \ s = \beta(e + p)$$

With EoM $0 = \partial_\mu T^{\mu\nu}$, div. of entropy current $S^\mu = su^\mu + O(\partial)$ can be evaluated as

$$\partial_\mu S^\mu = -T^{\mu\nu}_{(1)} \partial_\mu(\beta u_\nu) + O(\partial^3)$$

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2st law of thermodynamics says $\partial_\mu S^\mu \geq 0$, which is guaranteed **if RHS is expressed as a semi-positive bilinear** as

$$-T^{\mu\nu}_{(1)} \partial_\mu (\beta u_\nu) = \sum\nolimits_{X_i \in T_{(1)}} \lambda_i \, X_i^{\mu\nu} X_{i\,\nu\mu} \geq 0 \text{ with } \lambda_i \geq 0$$

(strong constraint !!)

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ex) heat current: $2h^{(\mu}u^{\nu)} \equiv h^\mu u^\nu + h^\nu u^\mu \in T^{\mu\nu}_{(1)} \ \ (u_\mu h^\mu = 0)$

$$\Rightarrow T^{\mu\nu}_{(1)} \partial_\mu(\beta u_\nu) = -\beta h^\mu \left(\beta \partial_{\perp\mu} \beta^{-1} + u^\nu \partial_\nu u^\mu\right) \geq 0$$

$$\Rightarrow h^\mu = -\kappa\left(\beta \partial_{\perp\mu} \beta^{-1} + u^\nu \partial_\nu u^\mu\right) \text{ with } \kappa \geq 0$$

Formulation of hydro w/o spin (3/3)

✓ **Constitutive relation up to 1st order w/o spin**

$$T^{\mu\nu}_{(0)} = e u^\mu u^\nu + p(g^{\mu\nu} + u^\mu u^\nu)$$

$$T^{\mu\nu}_{(1)} = -2\kappa\left(Du^{(\mu} + \beta\partial_\perp^{(\mu}\beta^{-1}\right)u^{\nu)} - 2\eta\partial_\perp^{<\mu}u^{\nu>} - \zeta(\partial_\mu u^\mu)\Delta^{\mu\nu}$$

heat current **shear viscosity** **bulk viscosity**

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✓ **Hydrodynamic equation w/o spin**

Hydrodynamic eq. = conservation law + constitutive relation

Euler eq.	$0 = \partial_\mu T^{\mu\nu}$	$T^{\mu\nu} = T^{\mu\nu}_{(0)}$
Navier-Stokes eq.	$0 = \partial_\mu T^{\mu\nu}$	$T^{\mu\nu} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)}$
⋮	⋮	⋮

Formulation of hydro with spin (1/4)

✓ **Strategy is the same**

✓ **Phenomenological formulation**

Step 1: Write down the conservation law

Step 2: Construct a constitutive relation

- define hydro variables
- write down all the possible tensor structures
- simplify the tensor structures by e.g. thermodynamics

Formulation of hydro with spin (2/4)

Step 1: Write down the conservation law

(1) energy conservation

$$0 = \partial_\mu T^{\mu\nu}$$

(canonical)

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$$\psi(x) \to S(\Lambda)\psi(\Lambda^{-1}x)$$

$$0 = \partial_\mu M^{\mu,\alpha\beta}$$

$$= \partial_\mu \left(L^{\mu,\alpha\beta} + \Sigma^{\mu,\alpha\beta}\right)$$

$$= \partial_\mu \left(x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha} + \Sigma^{\mu,\alpha\beta}\right)$$

cf. talk by Fukushima

$$\therefore \quad \partial_\mu \Sigma^{\mu,\alpha\beta} = T^{\alpha\beta} - T^{\beta\alpha}$$

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Consequence

(1) Spin must not be a hydro mode in a strict sense

cf. Hydro+ (talk by Stephanov)

(2) Nevertheless, it behaves *like* a hydro mode if $T^{\mu\nu}_{(a)} \ll 1$

➡ **inclusion of dissipative nature is crucially important**

Formulation of hydro with spin (3/4)

Step 2: Construct a constitutive relation for $T^{\mu\nu}, \Sigma^{\mu,\alpha\beta}$

(1) define hydro variables

4 DoGs

$$\{\beta, u^{\mu}\}$$

(2) simplify the tensor structure by thermodynamics

Formulation of hydro with spin (3/4)

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4 + 6 = 10 DoGs = # of EoMs

Introduce **spin chemical potential** $\{\beta, u^\mu, \boldsymbol{\omega^{\mu\nu}}\}$ with $\omega^{\mu\nu} = -\omega^{\nu\mu}$

✓ $\{\beta, u^\mu, \omega^{\mu\nu}\}$ are independent w/ each other at this stage ($\omega^{\mu\nu} \neq$ thermal vorticity)

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where I defined **spin density** $\sigma^{\alpha\beta}$

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Generalize **1st law of thermodynamics with spin** as

$$ds = \beta(de - \omega_{\mu\nu} d\sigma^{\mu\nu}), \;\; s = \beta(e + p - \omega_{\mu\nu}\sigma^{\mu\nu})$$

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$$\partial_{\mu} S^{\mu} = -T^{\mu\nu}_{(1s)} \frac{\partial_{\mu}(\beta u_{\nu}) + \partial_{\nu}(\beta u_{\mu})}{2} - T^{\mu\nu}_{(1a)} \left\{ \frac{\partial_{\mu}(\beta u_{\nu}) - \partial_{\nu}(\beta u_{\mu})}{2} - 2\beta\omega_{\mu\nu} \right\} + O(\partial^3)$$

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4 + 6 = 10 DoGs = # of EoMs

Introduce **spin chemical potential** $\{\beta, u^\mu, \boldsymbol{\omega}^{\mu\nu}\}$ with $\omega^{\mu\nu} = -\omega^{\nu\mu}$

- ✓ $\{\beta, u^\mu, \omega^{\mu\nu}\}$ are independent w/ each other at this stage ($\omega^{\mu\nu} \neq$ thermal vorticity)

(2) simplify the tensor structure by thermodynamics

Expand $T^{\mu\nu}, \Sigma^{\mu,\alpha\beta}$, i.t.o derivatives

$$T^{\mu\nu} = eu^\mu u^\nu + p(g^{\mu\nu} + u^\mu u^\nu) + T^{\mu\nu}_{(1)} + O(\partial^2), \quad \Sigma^{\mu,\alpha\beta} = u^\mu \sigma^{\alpha\beta} + O(\partial^1)$$

where I defined **spin density** $\sigma^{\alpha\beta}$

Generalize **1st law of thermodynamics with spin** as

$$ds = \beta(de - \omega_{\mu\nu} d\sigma^{\mu\nu}),\ \ s = \beta(e + p - \omega_{\mu\nu}\sigma^{\mu\nu})$$

With EoMs, div. of entropy current $S^\mu = su^\mu + O(\partial)$ can be evaluated as

$$\partial_\mu S^\mu = -T^{\mu\nu}_{(1s)} \frac{\partial_\mu(\beta u_\nu) + \partial_\nu(\beta u_\mu)}{2} - T^{\mu\nu}_{(1a)} \left\{ \frac{\partial_\mu(\beta u_\nu) - \partial_\nu(\beta u_\mu)}{2} - 2\beta\omega_{\mu\nu} \right\} + O(\partial^3)$$

- ✓ 2nd law of thermodynamics $\partial_\mu S^\mu \geq 0$ gives strong constraint on $T^{\mu\nu}_{(1)}$
- ✓ In global equilibrium $\partial_\mu S^\mu = 0$, so that ω = thermal vorticity.

Formulation of hydro with spin (4/4)

✓ Constitutive relation for $T^{\mu\nu}$ up to 1st order with spin

$$T^{\mu\nu}_{(0)} = eu^\mu u^\nu + p(g^{\mu\nu} + u^\mu u^\nu)$$

heat current shear viscosity bulk viscosity

$$T^{\mu\nu}_{(1)} = -2\kappa\left(Du^{(\mu} + \beta\partial_\perp^{(\mu}\beta^{-1}\right)u^{\nu)} - 2\eta\partial_\perp^{<\mu}u^{\nu>} - \zeta(\partial_\mu u^\mu)\Delta^{\mu\nu}$$

$$-2\lambda\left(-Du^{[\mu} + \beta\partial_\perp^{[\mu}\beta^{-1} + 4u_\rho\omega^{\rho[\mu}\right)u^{\nu]} - 2\gamma\left(\partial_\perp^{[\mu}u^{\nu]} - 2\Delta^\mu_\rho\Delta^\nu_\lambda\omega^{\rho\lambda}\right)$$

"boost heat current" "rotational (spinning) viscosity

NEW !

e.g. Eringen (1998); Lukaszewicz (1999)

✓ Relativistic generalization of a non-relativistic micropolar fluid

✓ "boost heat current" is a relativistic effect

✓ Hydrodynamics equation up to 1st order with spin

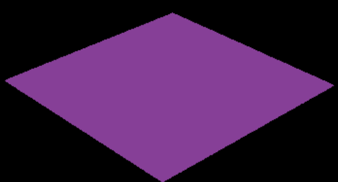
$$0 = \partial_\mu(T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)} + O(\partial^2)) \qquad \partial_\mu(u^\mu\sigma^{\alpha\beta}) = T^{\alpha\beta}_{(1)} - T^{\beta\alpha}_{(1)} + O(\partial^2)$$

Outline

1. ~~Introduction~~
2. ~~Formulation based on entropy-current analysis~~
3. **Linear mode analysis**
4. Summary

Linear mode analysis (1/2)

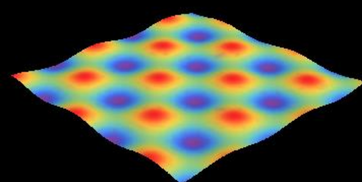
Setup: small perturbations on top of global therm. equilibrium



$\beta = \beta_0$

$u^\mu = (1, \mathbf{0})$

$\omega^{\mu\nu} = 0$



$\beta = \beta_0 + \delta\beta$

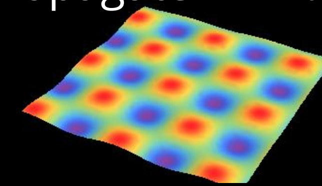
$u^\mu = (1, \mathbf{0}) + \delta u^\mu$

$\omega^{\mu\nu} = 0 + \delta\omega^{\mu\nu}$

$$i\partial_t \begin{pmatrix} \delta\beta \\ \delta u^\mu \\ \delta\omega^{\mu\nu} \end{pmatrix} = M \begin{pmatrix} \delta\beta \\ \delta u^\mu \\ \delta\omega^{\mu\nu} \end{pmatrix} + O(\delta^2)$$

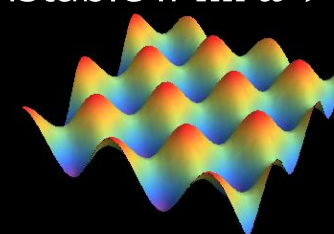
hydro evolution

propagate if $\mathrm{Im}\,\omega = 0$



dissipate if $\mathrm{Im}\,\omega < 0$

unstable if $\mathrm{Im}\,\omega > 0$



Linear mode analysis (2/2)

✓ **Hydro w/o spin** $\{\beta, u^\mu\}$

4 gapless modes

2 sound modes $\omega = \pm c_s k + O(k^2)$

2 shear modes $\omega = -i\dfrac{\eta k^2}{e+p} + O(k^4)$

where $c_s^2 \equiv \partial p/\partial e$

✓ **Hydro with spin** $\{\beta, u^\mu, \omega^{\mu\nu}\}$

Linear mode analysis (2/2)

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2 shear modes $\omega = -i\dfrac{\eta k^2}{e+p} + O(k^4)$

+ 6 dissipative gapped modes

3 "boost" modes $\omega = -2i\tau_b^{-1} + O(k^2)$

3 "spin" modes $\omega = -2i\tau_s^{-1} + O(k^2)$

where $\tau_s \equiv \dfrac{\partial\sigma^{ij}/\partial\omega^{ij}}{4\gamma}, \tau_b \equiv \dfrac{\partial\sigma^{i0}/\partial\omega^{i0}}{4\lambda}$

Linear mode analysis (2/2)

✓ Hydro w/o spin $\{\beta, u^\mu\}$

4 gapless modes

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✓ Hydro **with** spin $\{\beta, u^\mu, \omega^{\mu\nu}\}$

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where $\tau_{\mathrm{s}} \equiv \dfrac{\partial\sigma^{ij}/\partial\omega^{ij}}{4\gamma}, \tau_{\mathrm{b}} \equiv \dfrac{\partial\sigma^{i0}/\partial\omega^{i0}}{4\lambda}$

- ✓ We explicitly confirmed that **spin is dissipative**
- ✓ Time-scale of the dissipation is controlled by the new viscous constants γ, λ

Outline

1. ~~Introduction~~
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4. **Summary**

Summary

- ✓ Spin polarization in QGP is one of the hottest topics in HIC. But, its theory, in particular hydrodynamic framework, is still under construction
- ✓ **Relativistic spin hydrodynamics with 1st order dissipative corrections is formulated** for the first time based on the phenomenological entropy-current analysis
- ✓ **Spin must be dissipative** because of the mutual conversion between the orbital angular momentum and spin
- ✓ Linear mode analysis of the spin hydrodynamic equation also suggests that spin must be dissipative, whose time-scale is controlled by the new viscous constants γ, λ

Outlook: extension to 2nd order, Kubo formula, MHD, application to cond-mat, numerical simulations

BACKUP

Linearlized hydro eq.

$$M\delta\vec{c} = 0$$

where

$$M = \left(\begin{array}{c|ccccccc}
A_{3\times3} & \multicolumn{7}{c}{O} \\
\hline
 & -i\omega+(\gamma_\perp+\gamma')k_z^2 & +iD_sk_z & 0 & 0 & 0 & 0 & 0 \\
 & -2i\gamma' k_z & -i\omega+2D_s & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & -i\omega+(\gamma_\perp+\gamma')k_z^2 & -iD_sk_z & 0 & 0 & 0 \\
O & 0 & 0 & 2i\gamma' k_z & -i\omega+2D_s & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & -i\omega+2D_b & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & -i\omega+2D_b & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & -i\omega+2D_s
\end{array}\right)$$

$$A_{3\times3} = \begin{pmatrix} -i\omega+2c_s^2\lambda' k_z^2 & ik_z & -2iD_bk_z \\ ic_s^2k_z & -i\omega+\gamma_{\|}k_z^2 & 0 \\ 2ic_s^2\lambda' k_z & 0 & -i\omega+2D_b \end{pmatrix}$$

$$\delta\vec{c} \equiv (\delta\tilde{e}, \delta\tilde{\pi}^z, \delta\tilde{S}^{0z}, \delta\tilde{\pi}^x, \delta\tilde{S}^{zx}, \delta\tilde{\pi}^y, \delta\tilde{S}^{yz}, \delta\tilde{S}^{0x}, \delta\tilde{S}^{0y}, \delta\tilde{S}^{xy})^t$$

Dispersion relations

$$\omega = -2iD_s,$$

$$\omega = -2iD_b,$$

$$\omega = \begin{cases} -2iD_s - i\gamma' k_z^2 + \mathcal{O}(k_z^4), \\ -i\gamma_\perp k_z^2 + O(k_z^4),, \end{cases}$$

$$\omega = \begin{cases} \pm c_s k_z - i\frac{\gamma_\parallel}{2} k_z^2 + \mathcal{O}(k_z^3), \\ -2iD_b - 2ic_s^2 \lambda' k_z^2 + \mathcal{O}(k_z^4). \end{cases}$$

Further simplification by EoM

The 1st order constitutive relation reads

$$\Theta^{\mu\nu}_{(1s)} = 2h^{(\mu}u^{\nu)} + \tau^{\mu\nu}$$

$$\Theta^{\mu\nu}_{(1a)} = 2q^{[\mu}u^{\nu]} + \phi^{\mu\nu}$$

$$h^\mu = -\kappa(Du^\mu + \beta\partial^\mu_\perp T),$$

$$\tau^{\mu\nu} = -2\eta\partial^{\langle\mu}_\perp u^{\nu\rangle} - \zeta\theta\Delta^{\mu\nu},$$

$$q^\mu = -\lambda\big(-Du^\mu + \beta\partial^\mu_\perp T - 4\omega^{\mu\nu}u_\nu\big),$$

$$\phi^{\mu\nu} = -2\gamma\big(\partial^{[\mu}_\perp u^{\nu]} - 2\Delta^\mu_\rho\Delta^\nu_\lambda\omega^{\rho\lambda}\big),$$

By using LO hydro eq.,

$$(e+p)Du^\mu = -\partial^\mu_\perp p + \mathcal{O}(\partial^2)$$

we can further simplify *h,q* as

$$h^\mu = -\kappa\left[\frac{-\partial^\mu_\perp p}{e+p} + \beta\partial^\mu_\perp T + \mathcal{O}(\partial^2)\right] = \mathcal{O}(\partial^2),$$

$$q^\mu = -\lambda\left[\frac{2\partial^\mu_\perp p}{e+p} - 4\omega^{\mu\nu}u_\nu\right] + \mathcal{O}(\partial^2).$$