Hydrodynamics far from equilibrium: a concrete example

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New developments of hydrodynamics and its applications to heavy ion collisions

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What you will see:

✔ Motivation: why fluid-dynamical descriptions work in extreme conditions?

✔ Derivation of fluid dynamics from the Boltzmann equation: gradient expansion and method of moments

✔ Can we have hydrodynamic behavior far from equilibrium?
**Empirical:** Fluid-dynamical modeling of heavy ion collisions works well at RHIC and LHC energies

Main assumption: fluid dynamics can be applied at the very early stages – *Why?*
Validity of fluid dynamics traditionally associated with:

- “proximity” to (local) equilibrium
- “small” gradients

Separation of scales → macroscopic: $L$ microscopic: $\ell$

Knudsen number:

$$K_N \sim \frac{\ell}{L} \ll 1$$

Do these things occur early in HIC? No.
Extreme Conditions

Can this system be close to local equilibrium? Or domain of applicability of hydrodynamics better than expected?

Knudsen number is not small at early times

\[
Kn = \frac{\tau_\pi}{\nabla_\mu u^\mu}
\]

Knudsen number
Example:

Knudsen number in airplane aerodynamics

\[ \text{Kn} \ll 0.1 \]

Much smaller than what we get in HIC
Validity of fluid dynamics

Proximity to local equilibrium, small gradients
We can study this problem in Kinetic theory

Boltzmann eq.

\[ k^\mu \partial_\mu f_k = C[f] \]

Israel-Stewart-like theories

\[
\begin{align*}
\tau_\Pi \dot{\Pi} + \Pi &= -\zeta \theta + \ldots \\
\tau_\pi \dot{\pi}^{\mu\nu} + \pi^{\mu\nu} &= 2\eta \sigma^{\mu\nu} + \ldots
\end{align*}
\]
Basics of fluid dynamics

Effective theory describing the dynamics of a system over long-times and long-distances

Separation of scales $\rightarrow$ macroscopic: $L$  microscopic: $\ell$

Knudsen number: $K_N \sim \frac{\ell}{L} \ll 1$

Conservation laws + simple constitutive relations
Basics of fluid dynamics

Conservation Laws

\[ \partial_\mu T^{\mu\nu} = 0 \]
\[ \partial^\mu N_\mu = 0 \]

Tensor decomposition

\[ N^\mu = n u^\mu + n^\mu, \]
\[ T^{\mu\nu} = \varepsilon u^\mu u^\nu - \Delta^{\mu\nu} (P_0 + \Pi) + \pi^{\mu\nu} \]

Particle diffusion current
Bulk viscous pressure
Shear stress tensor

\[ \Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \]
Definition of “equilibrium state”

\[ \varepsilon \equiv u_\nu u_\mu T^{\mu \nu}, \quad \text{Definition of energy density} \]

\[ n \equiv u_\mu N^\mu. \quad \text{Definition of net-charge density} \]

\[ s_0 \equiv s_0 (n, \varepsilon), \quad \text{introduce an eq. entropy density} \]

\[ \beta_0 = \left. \frac{\partial s}{\partial \varepsilon} \right|_n, \quad p_0 = -\varepsilon + T_0 s_0 + \mu_0 n. \]

\[ \alpha_0 = \left. \frac{\partial s}{\partial n} \right|_s, \]

Definition of velocity

\[ u_\mu T^{\mu \nu} = \varepsilon u^\nu \quad \text{or} \quad N^\mu = nuu^\mu \quad \text{or} \ldots \]
Basics of fluid dynamics

Conservation Laws

\[ \partial_\mu T^{\mu\nu} = 0 \quad \partial^\mu N_\mu = 0 \]

Tensor decomposition

\[ N^\mu = n u^\mu + n^\mu, \]
\[ T^{\mu\nu} = \varepsilon u^\mu u^\nu - \Delta^{\mu\nu} (P_0 + \Pi) + \pi^{\mu\nu}? \]

Challenge: closing the equations
Closure: Gradient Expansion

our intuition on the validity of hydrodynamics comes mostly from this method
Gradient Expansion

\[ k^\mu \partial_\mu f_k = \frac{1}{\epsilon} C[f_k] \]

Knudsen number

Perturbative expansion

\[ f_k = f_k^{(0)} + \epsilon f_k^{(1)} + \epsilon^2 f_k^{(2)} + \ldots \]

local equilibrium

Result is a gradient expansion – more general than kinetic theory
Gradient Expansion

**1\text{st} order truncation: Navier-Stokes theory**

\[
\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}
\]
\[
\sigma^{\mu\nu} \equiv \frac{1}{2} (\nabla^{\mu} u^{\nu} + \nabla^{\nu} u^{\mu}) - \frac{1}{3} \Delta^{\mu\nu} \theta.
\]

**2\text{nd} order truncation: Burnett theory**

\[
\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \eta_1 \omega_{\lambda}^{(\mu} \omega^{\nu)\lambda} + \eta_2 \theta\sigma^{\mu\nu} + \eta_3 \sigma^{\lambda(\mu} \sigma_{\nu)\lambda} + \eta_4 \sigma_{\lambda}^{(\mu} \omega^{\nu)\lambda} + \eta_5 I^{(\mu I^{\nu)}
\]
\[
+ \eta_6 J^{(\mu J^{\nu)} + \eta_7 I^{(\mu J^{\nu)} + \eta_8 \nabla^{(\mu I^{\nu)} + \eta_9 \nabla^{(\mu J^{\nu).}
\]

\[
\omega^{\mu\nu} \equiv \frac{1}{2} (\nabla^{\mu} u^{\nu} - \nabla^{\nu} u^{\mu}).
\]
\[
\theta = \nabla_{\mu} u^{\mu},
\]
\[
I^{\mu} \equiv \nabla^{\mu} \alpha_0, \quad J^{\mu} \equiv \nabla^{\mu} \beta_0.
\]
Hydrodynamical constitutive equations are usually derived by *truncating* this series.

**Effective theory:** can be systematically corrected

Convergence is assumed!
Gradient Expansion Diverges (?)

H. Grad: CE is an asymptotic series, Physics of Fluids 6, 147 (1963).

First example of divergence: Couette flow problem (RTA), Santos et al, PRL 56, 1571 (1986).

Heller et al: Holography+Bjorken scaling, PRL 110, 211602 (2013) -- first time for an expanding system

Not necessarily a problem; but applicability and improvability not clear.
Example: Bjorken flow
Gradient Expansion Diverges (?)

Relativistic:

- Unstable equations of motion!
- Cannot be used for any practical application – order does not matter
Israel-Stewart theory is not obtained from this method

**Causality:** constitutive relations for the shear stress tensor cannot be imposed

\[
\Delta_{\alpha\beta}^{\mu\nu} D_{\pi^{\alpha\beta}}^{\mu\nu} + \frac{4}{3} \pi^{\mu\nu} \Theta + \frac{\pi^{\mu\nu}}{\tau_R} = 2 \frac{\eta}{\tau_R} \sigma^{\mu\nu}
\]

Contains *all orders* in gradients!
Matches gradient expansion up to second order
Method of moments and 14-moment approximation: basic ideas
Method of moments
H. Grad, Comm. Pure Appl. Math. 2, 331 (1949)

Expansion of $f(x, p)$ using a complete basis

$$f_k = f_{0k} + f_{0k} \tilde{f}_{0k} \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_{\ell}} \mathcal{H}_k^{(n\ell)} \rho_{(n)}^{\mu_1 \ldots \mu_\ell} k_{\langle \mu_1 \ldots k_{\mu_\ell} \rangle}.$$  

- truncation leads to hydro – no small parameter
- expansion in degrees of freedom
Israel-Stewart theory: *14-moment approximation*

\[ f_k = f_{0k} + f_{0k} \left( 1 - \alpha f_{0k} \right) \phi_k \]

- **equilibrium**
- **non-equilibrium**

1 – *Truncated* Taylor series in momentum

\[ \phi_k = \varepsilon + \varepsilon_{\mu} k^\mu + \varepsilon_{\mu\nu} k^\mu k^\nu \]

- degrees of freedom *reduced* by the **explicit truncation** of expansion!
- **14** fields left

Israel-Stewart theory: 14-moment approximation

\[ f_k = f_{0k} + f_{0k} (1 - \alpha f_{0k}) \phi_k \]

equilibrium  
non-equilibrium

2 – Expansion coefficients mapped to conserved currents via *matching conditions*

4 eqs.
\[
\begin{align*}
  u_\mu N^\mu &= n_0 \\
  u_\mu T^{\mu\nu} &= \varepsilon_0 u^\nu
\end{align*}
\]

definition of eq. state

10 eqs.
\[
\begin{align*}
  \pi^{\mu\nu} &= \Delta^{\mu\nu}_{\alpha\beta} T^{\alpha\beta} \\
  n^\mu &= \Delta^\mu_\alpha N^\alpha \\
  \Pi &= -\frac{1}{3} \Delta_{\mu\nu} T^{\mu\nu}
\end{align*}
\]

Israel-Stewart theory: \textit{14-moment approximation}

\[ f_k = f_{0k} + f_{0k} (1 - \alpha f_{0k}) \phi_k \]

\textbf{3 – Equations of motion taken from the second moment of the Boltzmann equation}

\[ \Delta_{\mu \nu}^\lambda \left( \partial_\alpha \int_K k^\alpha k^\mu k^\nu f_k = \int_K C[f] k^\mu k^\nu \right) \]
\[ u_\nu \Delta^\lambda_{\mu} \left( \partial_\alpha \int_K k^\alpha k^\mu k^\nu f_k = \int_K C[f] k^\mu k^\nu \right) \]
\[ u_\mu u_\nu \left( \partial_\alpha \int_K k^\alpha k^\mu k^\nu f_k = \int_K C[f] k^\mu k^\nu \right) \]

Final Equations of motion
GSD et al, PRD 85, 114047 (2012)

\[ \dot{\Pi} = -\frac{\Pi}{\tau_\Pi} - \beta_\Pi \theta - \ell_\Pi n \partial \cdot n - \tau_\Pi n n \cdot \dot{u} - \delta_\Pi \Pi \Pi \theta \]
\[ -\lambda_\Pi n n \cdot \nabla \alpha_0 + \lambda_\Pi \pi^{\mu\nu} \sigma_{\mu\nu}, \quad (20) \]

\[ \dot{n}^{\mu} = -\frac{n^{\mu}}{\tau_n} + \beta_n \nabla^{\mu} \alpha_0 - n_\nu \omega^{\nu\mu} - \delta_{nn} n^{\mu} \theta - \ell_{n\Pi} \nabla^{\mu} \Pi \]
\[ + \ell_{n\pi} \Delta^{\mu\nu} \partial_\lambda \pi^{\lambda}_\nu + \tau_{n\Pi} \Pi \dot{u}^{\mu} - \tau_{n\pi} \pi^{\mu}_\nu \dot{u}^{\nu} \]
\[ -\lambda_{nn} n^{\nu} \sigma^{\mu}_\nu + \lambda_{n\Pi} \Pi \nabla^{\mu} \alpha_0 - \lambda_{n\pi} \pi^{\mu\nu} \nabla^{\nu} \alpha_0, \quad (21) \]

\[ \dot{\pi}^{\mu\nu} = -\frac{\pi^{\mu\nu}}{\tau_\pi} + 2\beta_\pi \sigma^{\mu\nu} + 2\pi^{\mu}_\alpha \omega^{\nu}_\alpha - \tau_{\pi n} n^{\mu} \dot{u}^{\nu} \]
\[ + \ell_{\pi n} \nabla^{\mu} n^{\nu} - \delta_{\pi\pi} \pi^{\mu\nu} \theta - \tau_{\pi\pi} \pi^{\mu}_\alpha \sigma^{\nu}_\alpha \]
\[ + \lambda_{\pi n} n^{\mu} \nabla^{\nu} \alpha_0 + \lambda_{\pi\Pi} \Pi \sigma^{\mu\nu}. \quad (22) \]

Many terms originally omitted by Israel and Stewart.

• Nontrivial assumption: application of matching conditions

• This step does not require proximity to eq.

• All previous steps can be applied assuming the form: $f_{0k} \left( \lambda, u_\mu k^\mu / \Lambda \right)$

$$f_k = f_{0k} + \delta f_k$$

- isotropic, non-equilibrium correction, anisotropic scalars
Matching conditions

\[ f_{k} = f_{0k} \left( \lambda, u_{\mu} k^{\mu} / \Lambda \right) + \delta f_{k} \]

5 parameters – can be associated with velocity, energy density and particle density

\[
\begin{align*}
\lambda &= \lambda (n, \varepsilon) \\
\Lambda &= \Lambda (n, \varepsilon)
\end{align*}
\]

14-moment approx.: shear term only

\[ f_{k} = f_{0k} + \frac{1}{2 I_{42}} \pi^{\mu\nu} k_{\mu} k_{\nu} \]

\[ I_{42} = \frac{1}{15} \int \frac{d^{3} k}{(2\pi)^{3} k^{0}} |k|^{4} f_{0k} \]
Equations of motion: ultrarelativistic gas of hard spheres

We recover the usual equation for the shear stress:

\[
\dot{\pi}^{\mu\nu} + \frac{\pi^{\mu\nu}}{\tau_\pi} - \frac{2I_{40}}{3I_{50}} \sigma^{\rho\lambda} \pi^{\mu\lambda} = 2 \frac{\eta}{\tau_\pi} \sigma^{\mu\nu} - 2 \sigma^{\nu\lambda} \pi^{\mu \lambda} - \frac{4}{3} \pi^{\mu\nu} \theta
\]

Transport coefficients: functional dependence on \( f_{0k} \)

\[
\frac{1}{\tau_\pi} = \left( 1 + 4 \frac{P_0 I_{40}}{n_0 I_{50}} \right) \frac{1}{3 \ell_{\text{mfp}}} \\
\eta = \frac{4I_{40} \varepsilon^2}{3n_0 I_{50} + 12P_0 I_{40}} \ell_{\text{mfp}}
\]

Thermodynamic integrals:

\[
I_{nq} = \frac{(-1)^q}{(2q + 1)!!} \int dK E_k^{n-2q} (\Delta^\alpha k^\alpha k^\beta)^q f_{0k}
\]
Equations of motion: ultrarelativistic gas of hard spheres

We recover the usual equation for the shear stress:

\[
\dot{\pi} \langle \mu \nu \rangle + \frac{\pi^{\mu \nu}}{\tau_\pi} - \frac{2I_{40}}{3I_{50}} \sigma_\pi \chi^{\mu \nu} - \frac{2}{3} \sigma_\pi \chi^{\mu \nu} \chi = 2 \frac{\eta}{\tau_\pi} \sigma^{\mu \nu} - 2 \sigma_\chi^{\mu \nu} \chi - \frac{4}{3} \pi^{\mu \nu} \theta
\]

“Equilibrium” Transport coefficients:

\[
\tau_\pi = \frac{9}{5} \ell_{\text{mfp}} \\
\eta = \frac{6}{5} \frac{T}{\sigma}
\]

Coefficients derived by Israel-Stewart
Example of non-equilibrium state: “over-occupied” state

\[ f_{0k} = \lambda \Theta (\Lambda - E_k) \]

Matching:
\[ \Lambda = 4T \]
\[ \lambda \sim \exp(\alpha) \]
Equations of motion: ultrarelativistic gas of hard spheres

We recover the usual equation for the shear stress:

\[ \dot{\pi} \langle \mu \nu \rangle + \frac{\pi^{\mu \nu}}{\tau_\pi} - \frac{2I_{40}}{3I_{50}} \sigma_{\pi \lambda} \langle \mu \pi \nu \lambda \rangle = 2 \frac{\eta}{\tau_\pi} \sigma^{\mu \nu} - 2 \sigma^{\mu \nu} \lambda - \frac{4}{3} \pi^{\mu \nu} \theta \]

Over-occupied Transport coefficients:

\[ \tau_\pi = \frac{18}{13} \ell_{\text{mfp}} \]

\[ \eta = \frac{84}{65} \frac{T}{\sigma_T} \]

- qualitatively the same
- appears to be slightly more viscous
Equations of motion: ultrarelativistic gas of hard spheres

We recover the usual equation for the shear stress:

\[
\dot{\pi}^{\mu \nu} + \frac{\pi^{\mu \nu}}{\tau} - \frac{2I_{40}}{3I_{50}} \sigma^{\pi \lambda} \dot{\pi}^{\mu \nu} = 2 \frac{\eta}{\tau} \sigma^{\mu \nu} - 2 \sigma^{\lambda} \dot{\pi}^{\mu \nu} \lambda - \frac{4}{3} \pi^{\mu \nu} \theta
\]

<table>
<thead>
<tr>
<th>( f_{0k} )</th>
<th>( \lambda \exp (-E_k / \Lambda) )</th>
<th>( \lambda \Theta (\Lambda - E_k) )</th>
<th>( \lambda \delta (E_k - \Lambda) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tau_\pi )</td>
<td>( \frac{9}{5} \ell_{mfp} )</td>
<td>( \frac{18}{13} \ell_{mfp} )</td>
<td>( \frac{9}{7} \ell_{mfp} )</td>
</tr>
<tr>
<td>( \eta )</td>
<td>( \frac{6}{5} \frac{T}{\sigma_T} )</td>
<td>( \frac{84}{65} \frac{T}{\sigma_T} )</td>
<td>( \frac{9}{7} \frac{T}{\sigma_T} )</td>
</tr>
</tbody>
</table>

Coefficients do not change much with \( f_{0k} \). Can we see this?
Simulation: Boltzmann eq. + Bjorken flow

.ultrarelativistic classical gas of hard spheres

Initial conditions, fixed energy

shear viscosity

\[ \frac{\eta}{n} \approx 6 \]

Evolution of shear stress does not see this non-equilibrium effect

*not* in equilibrium!
Is it ok to treat anisotropic part as a correction?

\[ f_k = f_{0k} + f_{0k} (1 - \alpha f_{0k}) \phi_k \]

**Isotropic**  
**Anisotropic corrections**

This appears to be a good approximation so far ...  
(checked in Gubser or Bjorken flow)

Otherwise, one gets anisotropic hydrodynamics
Conclusions

- The applicability of fluid-dynamical models of heavy ion collisions cannot be easily justified.

- The derivation of hydrodynamics using the method of moments is more general than previously considered: **hydrodynamic equations can be obtained even expanding around far from equilibrium.**