INSTITUTO DE FÍSICA
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Hydrodynamics far from equilibrium: a concrete example

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New developments of hydrodynamics and its applications to heavy ion collisions

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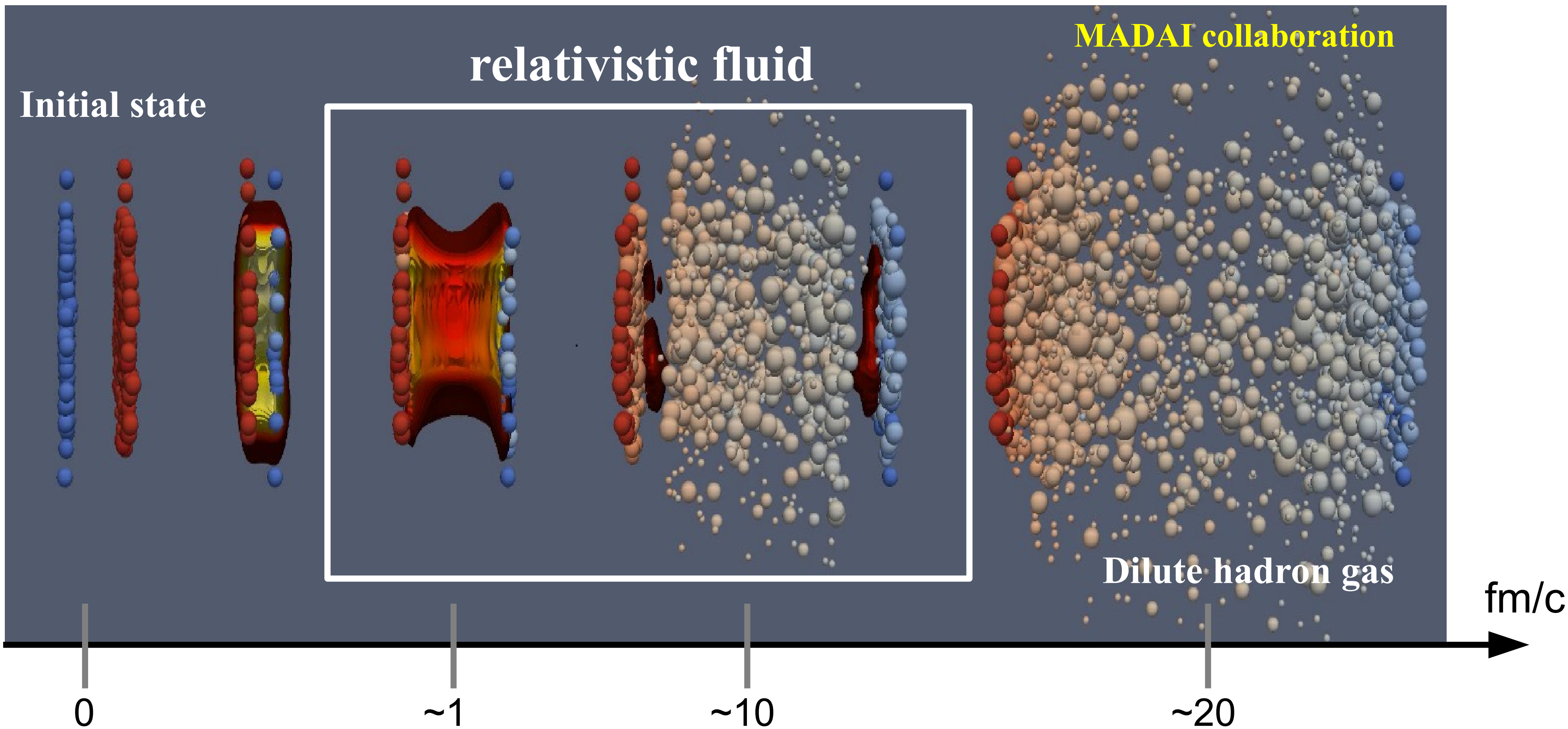
<u>What you will see:</u>

- ✔ Motivation: why fluid-dynamical descriptions work in extreme conditions?

- ✔ Derivation of fluid dynamics from the Boltzmann equation: *<u>gradient expansion</u>* and *<u>method of moments</u>*

- ✔ Can we have hydrodynamic behavior far from equilibrium?

Empirical: Fluid-dynamical modeling of heavy ion collisions works well at RHIC and LHC energies



Main assumption: fluid dynamics can be applied at the very early stages – *Why?*

Validity of fluid dynamics traditionally associated with:

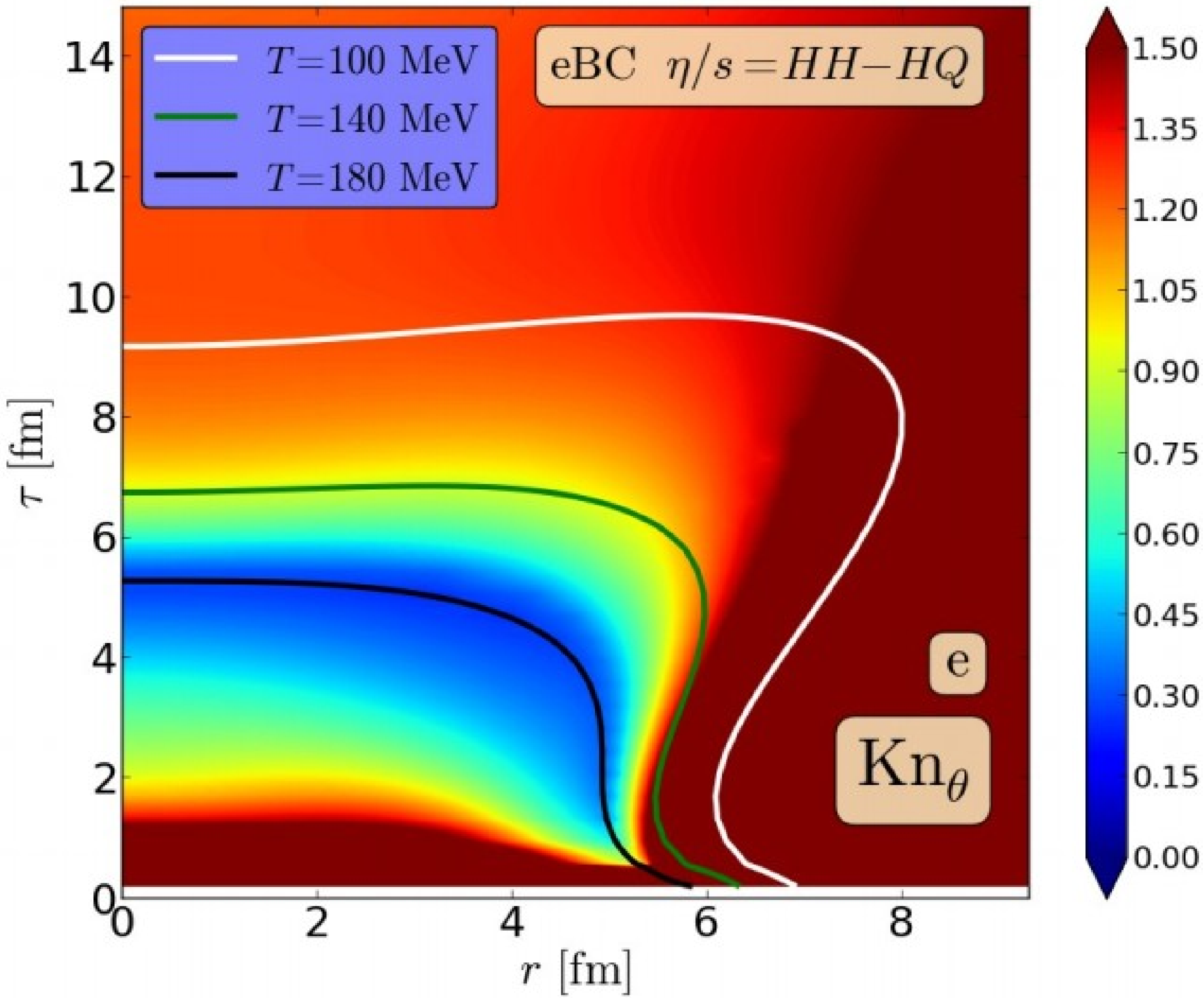
- **"proximity" to (local) equilibrium**
- **"small" gradients**

Separation of scales $\longrightarrow$ <u>**macroscopic:**</u> L <u>**microscopic:**</u> ℓ

Knudsen number: $K_N \sim \frac{\ell}{L} \ll 1$

Do these things occur early in HIC? No.

Extreme Conditions



Knudsen number

$$\mathrm{Kn} = \tau_\pi \nabla_\mu u^\mu$$

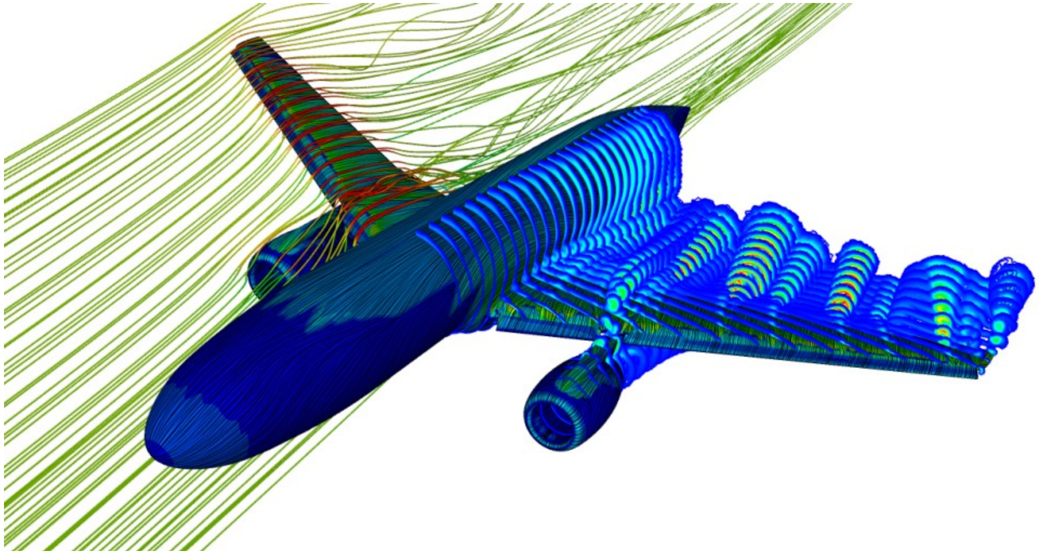
is not small at early times

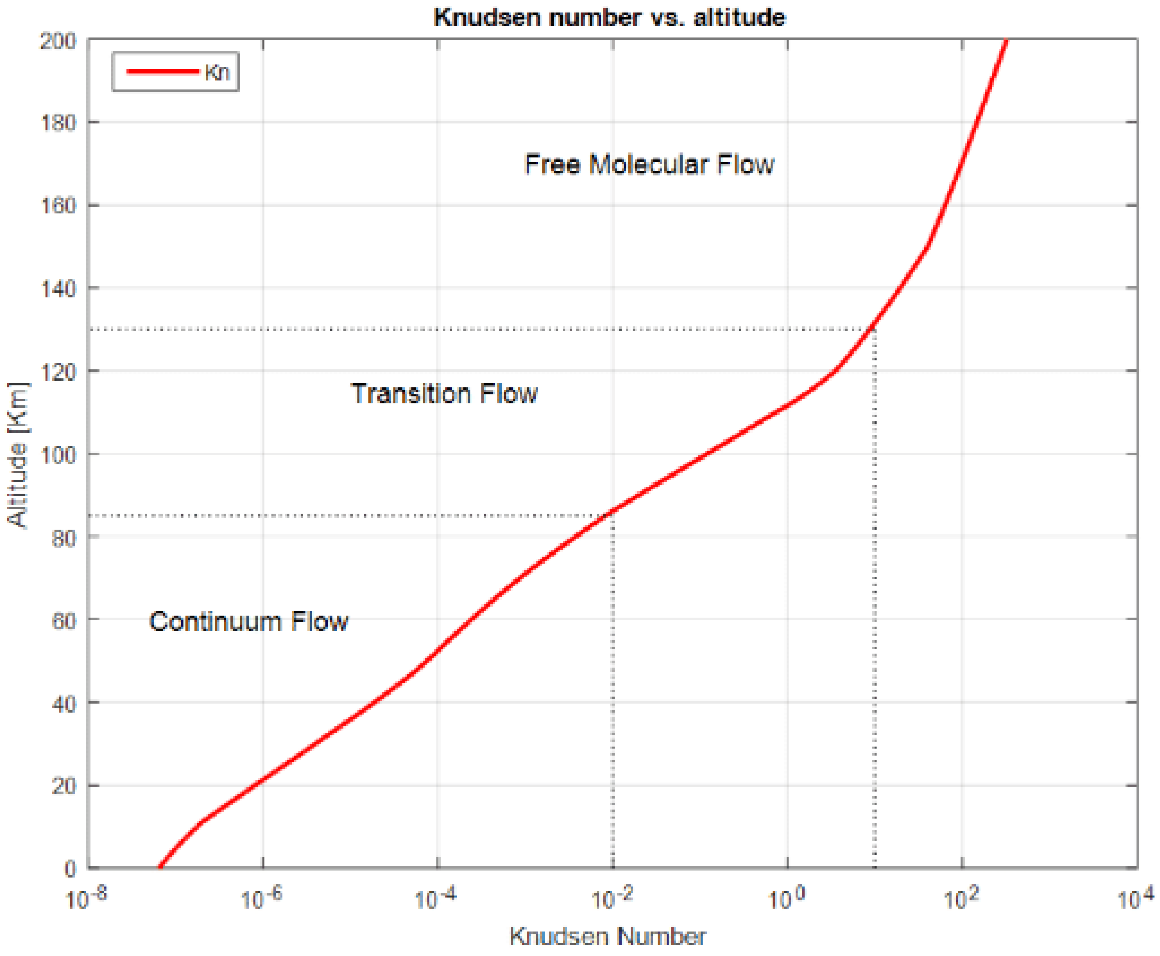
Can this system be close to local equilibrium?

Or domain of applicability of hydrodynamics better than expected?

Example:

Knudsen number in airplane aerodynamics





$Kn << 0.1$

Much smaller than what we get in HIC

Validity of fluid dynamics

???

Proximity to local equilibrium, small gradients

We can study this problem in Kinetic theory

$$k^{\mu}\partial_{\mu}f_{\mathbf{k}} = C[f]$$

Boltzmann eq.

????

$$\tau_{\Pi}\dot{\Pi} + \Pi = -\zeta\theta + \quad ...$$

$$\tau_{\pi}\dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \quad ...$$

Israel-Stewart-like theories

Basics of fluid dynamics

Effective theory describing the dynamics
of a system over long-times and long-distances

Separation of scales → **macroscopic:** L **microscopic:** ℓ

Knudsen number: $$K_N \sim \frac{\ell}{L} \ll 1$$

Conservation laws

+

simple constitutive relations

Basics of fluid dynamics

Conservation Laws

$$\partial_\mu T^{\mu\nu} = 0 \qquad \partial^\mu N_\mu = 0$$

Tensor decomposition

$$N^\mu = n u^\mu + n^\mu,$$
$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - \Delta^{\mu\nu}(P_0 + \Pi) + \pi^{\mu\nu}$$

Particle diffusion current (n^μ)

Bulk viscous pressure (Π)

Shear stress tensor ($\pi^{\mu\nu}$)

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$$

Definition of “equilibrium state”

$$\varepsilon \equiv u_\nu u_\mu T^{\mu\nu},$$ → Definition of **energy density**

$$n \equiv u_\mu N^\mu.$$ → Definition of **net-charge density**

$s_0 \equiv s_0(n, \varepsilon)$, introduce an eq. entropy density

$$\beta_0 = \left.\frac{\partial s}{\partial \varepsilon}\right|_n, \qquad \alpha_0 = \left.\frac{\partial s}{\partial n}\right|_s,$$

$$p_0 = -\varepsilon + T_0 s_0 + \mu_0 n.$$

Definition of velocity

$u_\mu T^{\mu\nu} = \varepsilon u^\nu$ or $N^\mu = n u^\mu$ or ...

Basics of fluid dynamics

Conservation Laws

$$\partial_\mu T^{\mu\nu} = 0 \qquad \partial^\mu N_\mu = 0$$

Tensor decomposition

$$N^\mu = n u^\mu + \underline{n^\mu}^{?},$$

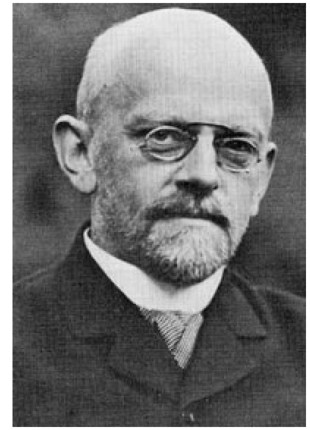
$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - \Delta^{\mu\nu}\left(P_0 + \underline{\Pi}^{?}\right) + \underline{\pi^{\mu\nu}}^{?}$$

Challenge: closing the equations

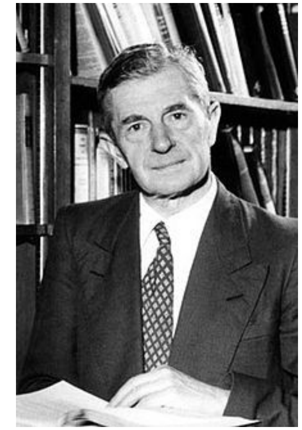
Closure: Gradient Expansion

our intuition on the validity of hydrodynamics comes mostly from this method

Gradient Expansion



Hilbert



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Enskog

$$k^{\mu}\partial_{\mu} f_{\mathbf{k}} = \frac{1}{\epsilon} C\left[f_{\mathbf{k}}\right]$$

ϵ → **Knudsen number**

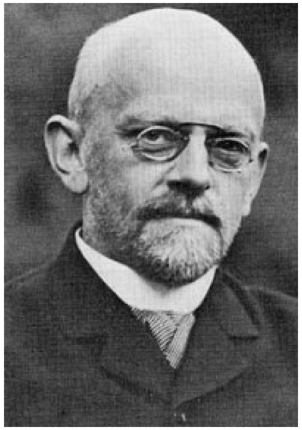
Perturbative expansion

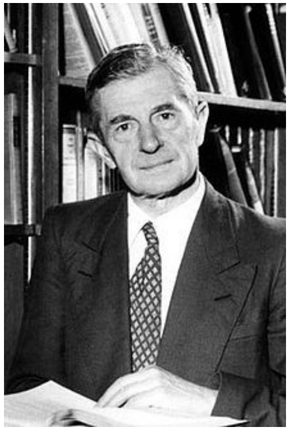
$$f_{\mathbf{k}} = f_{\mathbf{k}}^{(0)} + \epsilon f_{\mathbf{k}}^{(1)} + \epsilon^{2} f_{\mathbf{k}}^{(2)} + \ldots$$

$f_{\mathbf{k}}^{(0)}$ ← **local equilibrium**

Result is a gradient expansion – more general than kinetic theory

Gradient Expansion


Hilbert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1st order truncation: Navier-Stokes theory

$$\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}$$

$$\sigma^{\mu\nu} \equiv \frac{1}{2}(\nabla^\mu u^\nu + \nabla^\nu u^\mu) - \frac{1}{3}\Delta^{\mu\nu}\theta.$$

2nd order truncation: Burnett theory

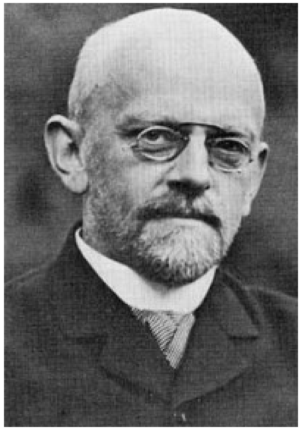
$$\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \eta_1\omega_\lambda^{\;\langle\mu}\omega^{\nu\rangle\lambda} + \eta_2\theta\sigma^{\mu\nu} + \eta_3\sigma^{\lambda\langle\mu}\sigma_\lambda^{\;\nu\rangle} + \eta_4\sigma_\lambda^{\;\langle\mu}\omega^{\nu\rangle\lambda} + \eta_5 I^{\langle\mu}I^{\nu\rangle} + \eta_6 J^{\langle\mu}J^{\nu\rangle} + \eta_7 I^{\langle\mu}J^{\nu\rangle} + \eta_8\nabla^{\langle\mu}I^{\nu\rangle} + \eta_9\nabla^{\langle\mu}J^{\nu\rangle}.$$

$$\omega^{\mu\nu} \equiv \frac{1}{2}(\nabla^\mu u^\nu - \nabla^\nu u^\mu).$$

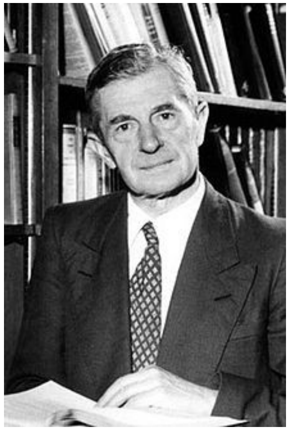
$$\theta = \nabla_\mu u^\mu,$$

$$I^\mu \equiv \nabla^\mu\alpha_0, \quad J^\mu \equiv \nabla^\mu\beta_0,$$

Gradient Expansion



Hilbert



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Second-order truncation: Burnett theory

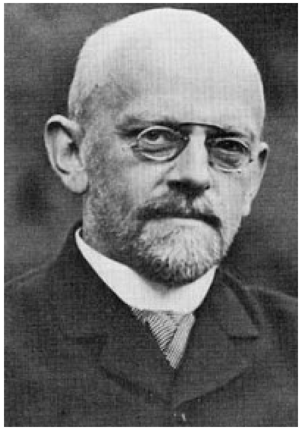
$$\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \eta_1 \omega_{\lambda}^{\ \langle\mu} \omega^{\nu\rangle\lambda} + \eta_2 \theta \sigma^{\mu\nu} + \eta_3 \sigma^{\lambda\langle\mu} \sigma_{\lambda}^{\ \nu\rangle} + \eta_4 \sigma_{\lambda}^{\ \langle\mu} \omega^{\nu\rangle\lambda} + \eta_5 I^{\langle\mu} I^{\nu\rangle} + \eta_6 J^{\langle\mu} J^{\nu\rangle} + \eta_7 I^{\langle\mu} J^{\nu\rangle} + \eta_8 \nabla^{\langle\mu} I^{\nu\rangle} + \eta_9 \nabla^{\langle\mu} J^{\nu\rangle}.$$

Hydrodynamical constitutive equations are usually derived by *truncating* this series.

Effective theory: can be systematically corrected

Convergence is assumed!

Gradient Expansion Diverges (?)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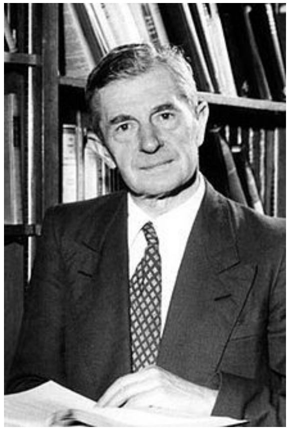
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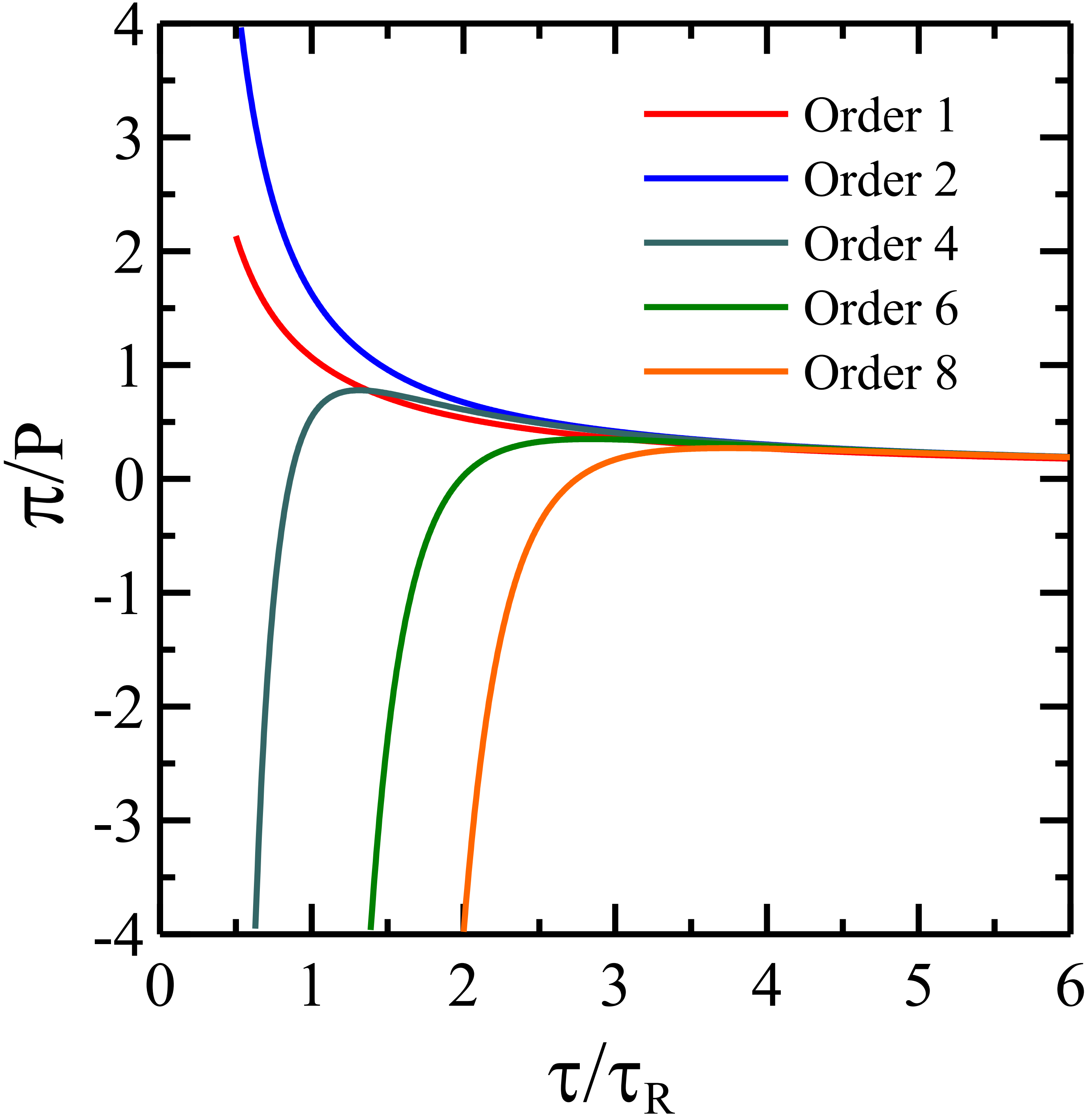
H. Grad: CE is an asymptotic series, Physics of Fluids 6, 147 (1963).

First example of divergence: Couette flow problem (RTA), Santos *et al*, PRL 56, 1571 (1986).

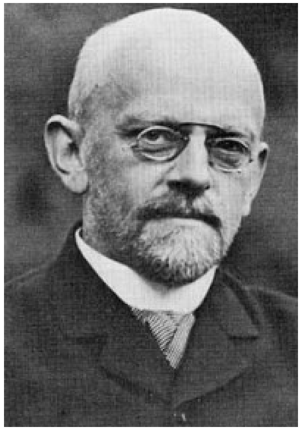
Heller *et al*: Holography+Bjorken scaling, PRL 110, 211602 (2013) -- first time for an expanding system

Not necessarily a problem; but applicability and improvability not clear.

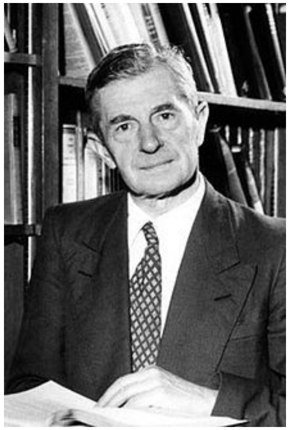
Example: Bjorken flow



Gradient Expansion Diverges (?)



Hilbert



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Relativistic:

- Unstable equations of motion!

- Cannot be used for any practical application – order does not matter

Israel-Stewart theory is not obtained from this method

Causality: constitutive relations for the shear stress tensor cannot be imposed

Dynamical equation, e.g. Israel-Stewart theory

$$\Delta^{\mu\nu}_{\alpha\beta} D\pi^{\alpha\beta} + \frac{4}{3}\pi^{\mu\nu}\Theta + \frac{\pi^{\mu\nu}}{\tau_R} = 2\frac{\eta}{\tau_R}\sigma^{\mu\nu}$$

Contains *all orders* in gradients!

Matches gradient expansion up to second order

Method of moments and 14-moment approximation: basic ideas

Method of moments

H. Grad, Comm. Pure Appl. Math. 2, 331 (1949)

H. Grad

$$f(\mathbf{x},\mathbf{p})$$

Expansion of f(**x**,**p**) using a complete basis

does not have to be eq.

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}} \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} \mathcal{H}_{\mathbf{k}}^{(n\ell)} \rho_{(n)}^{\mu_1 \ldots \mu_\ell} k_{\langle \mu_1} \cdots k_{\mu_\ell \rangle} \,.$$

- truncation leads to hydro – *no small parameter*
- expansion in *degrees of freedom*

Israel-Stewart theory: *14-moment approximation*

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + f_{0\mathbf{k}}\left(1 - a f_{0\mathbf{k}}\right)\phi_{\mathbf{k}}$$

equilibrium *non-equilibrium*

1 – ***Truncated*** Taylor series in momentum

$$\phi_{\mathbf{k}} = \varepsilon + \varepsilon_{\mu} k^{\mu} + \varepsilon_{\mu\nu} k^{\mu} k^{\nu}$$

- degrees of freedom *reduced* by the **explicit truncation** of expansion!
- *14* fields left

W. Israel & J M.Stewart, Ann. Phys. (N.Y.) 118, 341 (1979).

Israel-Stewart theory: *14-moment approximation*

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + f_{0\mathbf{k}}\left(1 - a f_{0\mathbf{k}}\right)\phi_{\mathbf{k}}$$

equilibrium ($f_{0\mathbf{k}}$) *non-equilibrium* ($\phi_{\mathbf{k}}$)

2 – Expansion coefficients mapped to conserved currents via ***matching conditions***

4 eqs.

$$u_\mu N^\mu = n_0$$
$$u_\mu T^{\mu\nu} = \varepsilon_0 u^\nu$$

definition of eq. state

10 eqs.

$$\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} T^{\alpha\beta}$$
$$n^\mu = \Delta^\mu_\alpha N^\alpha$$
$$\Pi = -\frac{1}{3}\Delta_{\mu\nu} T^{\mu\nu}$$

W. Israel & J M.Stewart, Ann. Phys. (N.Y.) 118, 341 (1979).

Israel-Stewart theory: *14-moment approximation*

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + f_{0\mathbf{k}}\left(1 - a f_{0\mathbf{k}}\right)\phi_{\mathbf{k}}$$

equilibrium ($f_{0\mathbf{k}}$) *non-equilibrium* ($\phi_{\mathbf{k}}$)

3 – Equations of motion taken from the ***second moment*** of the Boltzmann equation

$$\Delta^{\lambda\rho}_{\mu\nu}\left(\partial_\alpha \int_K k^\alpha k^\mu k^\nu f_{\mathbf{k}} = \int_K C[f]\, k^\mu k^\nu\right) \longleftrightarrow \text{shear}$$

$$u_\nu \Delta^{\lambda}_{\mu}\left(\partial_\alpha \int_K k^\alpha k^\mu k^\nu f_{\mathbf{k}} = \int_K C[f]\, k^\mu k^\nu\right) \longleftrightarrow \text{diffusion}$$

$$u_\mu u_\nu \left(\partial_\alpha \int_K k^\alpha k^\mu k^\nu f_{\mathbf{k}} = \int_K C[f]\, k^\mu k^\nu\right) \longleftrightarrow \text{bulk}$$

W. Israel & J M.Stewart, Ann. Phys. (N.Y.) 118, 341 (1979).

Final Equations of motion

GSD et al, PRD 85, 114047 (2012)

$$\dot{\Pi} = -\frac{\Pi}{\tau_\Pi} - \beta_\Pi \theta - \ell_{\Pi n} \partial \cdot n - \tau_{\Pi n} n \cdot \dot{u} - \delta_{\Pi\Pi} \Pi \theta - \lambda_{\Pi n} n \cdot \nabla \alpha_0 + \lambda_{\Pi \pi} \pi^{\mu\nu} \sigma_{\mu\nu} \ , \tag{20}$$

$$\dot{n}^{\langle\mu\rangle} = -\frac{n^\mu}{\tau_n} + \beta_n \nabla^\mu \alpha_0 - n_\nu \omega^{\nu\mu} - \delta_{nn} n^\mu \theta - \ell_{n\Pi} \nabla^\mu \Pi + \ell_{n\pi} \Delta^{\mu\nu} \partial_\lambda \pi_\nu^\lambda + \tau_{n\Pi} \Pi \dot{u}^\mu - \tau_{n\pi} \pi_\nu^\mu \dot{u}^\nu - \lambda_{nn} n^\nu \sigma_\nu^\mu + \lambda_{n\Pi} \Pi \nabla^\mu \alpha_0 - \lambda_{n\pi} \pi^{\mu\nu} \nabla_\nu \alpha_0 \ , \tag{21}$$

$$\dot{\pi}^{\langle\mu\nu\rangle} = -\frac{\pi^{\mu\nu}}{\tau_\pi} + 2\beta_\pi \sigma^{\mu\nu} + 2\pi_\alpha^{\langle\mu} \omega^{\nu\rangle\alpha} - \tau_{\pi n} n^{\langle\mu} \dot{u}^{\nu\rangle} + \ell_{\pi n} \nabla^{\langle\mu} n^{\nu\rangle} - \delta_{\pi\pi} \pi^{\mu\nu} \theta - \tau_{\pi\pi} \pi_\alpha^{\langle\mu} \sigma^{\nu\rangle\alpha} + \lambda_{\pi n} n^{\langle\mu} \nabla^{\nu\rangle} \alpha_0 + \lambda_{\pi\Pi} \Pi \sigma^{\mu\nu} \ . \tag{22}$$

Many terms originally omitted by Israel and Stewart.

W. Israel & J M.Stewart, Ann. Phys. (N.Y.) 118, 341 (1979).

- Nontrivial assumption: application of matching conditions

- This step does <u>*not*</u> require proximity to eq.

- *All previous steps can be applied* ***assuming the form:*** $f_{0\mathbf{k}}\left(\lambda, u_\mu k^\mu / \Lambda\right)$

scalars

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + \delta f_{\mathbf{k}}$$

isotropic, non-equilibrium

correction, anisotropic

Matching conditions

$$f_{\mathbf{k}} = f_{0\mathbf{k}}\left(\lambda, u_{\mu}k^{\mu}/\Lambda\right) + \delta f_{\mathbf{k}}$$

5 parameters – can be associated with velocity, energy density and particle density

$$\begin{cases} \lambda = \lambda\left(n, \varepsilon\right) \\ \Lambda = \Lambda\left(n, \varepsilon\right) \end{cases}$$

14-moment approx.: shear term only

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + \frac{1}{2I_{42}}\pi^{\mu\nu}k_{\mu}k_{\nu}$$

$$I_{42} = \frac{1}{15}\int \frac{d^3k}{(2\pi)^3 k^0}\left|\mathbf{k}\right|^4 f_{0\mathbf{k}}$$

Equations of motion: *ultrarelativistic* gas of *hard spheres*

We recover the usual equation for the shear stress:

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} - \frac{2I_{40}}{3I_{50}}\sigma\pi^{\lambda\langle\mu}\pi_\lambda^{\nu\rangle} = 2\frac{\eta}{\tau_\pi}\sigma^{\mu\nu} - 2\sigma_\lambda^{\langle\mu}\pi^{\nu\rangle\lambda} - \frac{4}{3}\pi^{\mu\nu}\theta$$

Transport coefficients:

functional dependence on $f_{0\mathbf{k}}$

$$\frac{1}{\tau_\pi} = \left(1 + 4\frac{P_0 I_{40}}{n_0 I_{50}}\right)\frac{1}{3\ell_{\mathrm{mfp}}}$$

$$\eta = \frac{4I_{40}\varepsilon^2}{3n_0 I_{50} + 12P_0 I_{40}}\ell_{\mathrm{mfp}}$$

Thermodynamic integrals: $I_{nq} = \frac{(-1)^q}{(2q+1)!!}\int dK E_{\mathbf{k}}^{n-2q}\left(\Delta^{\alpha\beta}k_\alpha k_\beta\right)^q f_{0\mathbf{k}}$

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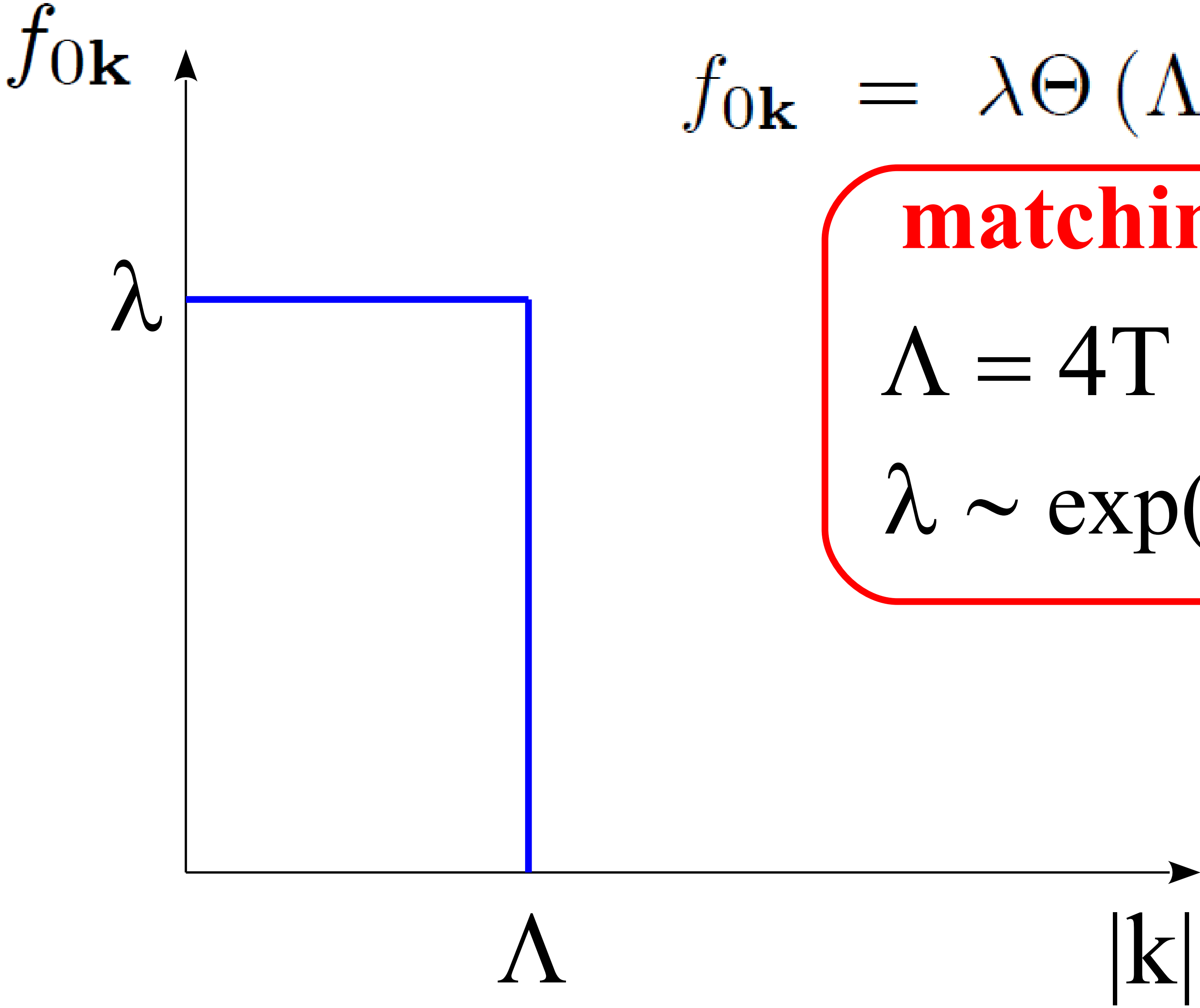
"Equilibrium" Transport coefficients:

$$\tau_\pi = \frac{9}{5}\ell_{\mathrm{mfp}}$$

$$\eta = \frac{6}{5}\frac{T}{\sigma}$$

Coefficients derived by Israel-Stewart

Example of non-equilibrium state: “over-occupied” state



$$f_{0\mathbf{k}} = \lambda\Theta\left(\Lambda - E_{\mathbf{k}}\right)$$

matching

$\Lambda = 4\mathrm{T}$

$\lambda \sim \exp(\alpha)$

Equations of motion: *ultrarelativistic* gas of *hard spheres*

We recover the usual equation for the shear stress:

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} - \frac{2I_{40}}{3I_{50}}\sigma\pi^{\lambda\langle\mu}\pi_\lambda^{\nu\rangle} = 2\frac{\eta}{\tau_\pi}\sigma^{\mu\nu} - 2\sigma_\lambda^{\langle\mu}\pi^{\nu\rangle\lambda} - \frac{4}{3}\pi^{\mu\nu}\theta$$

Over-occupied Transport coefficients:

$$\tau_\pi = \frac{18}{13}\ell_{\mathrm{mfp}}$$

$$\eta = \frac{84}{65}\frac{T}{\sigma_T}$$

- qualitatively the same
- appears to be slightly more viscous

Equations of motion: *ultrarelativistic* gas of *hard spheres*

We recover the usual equation for the shear stress:

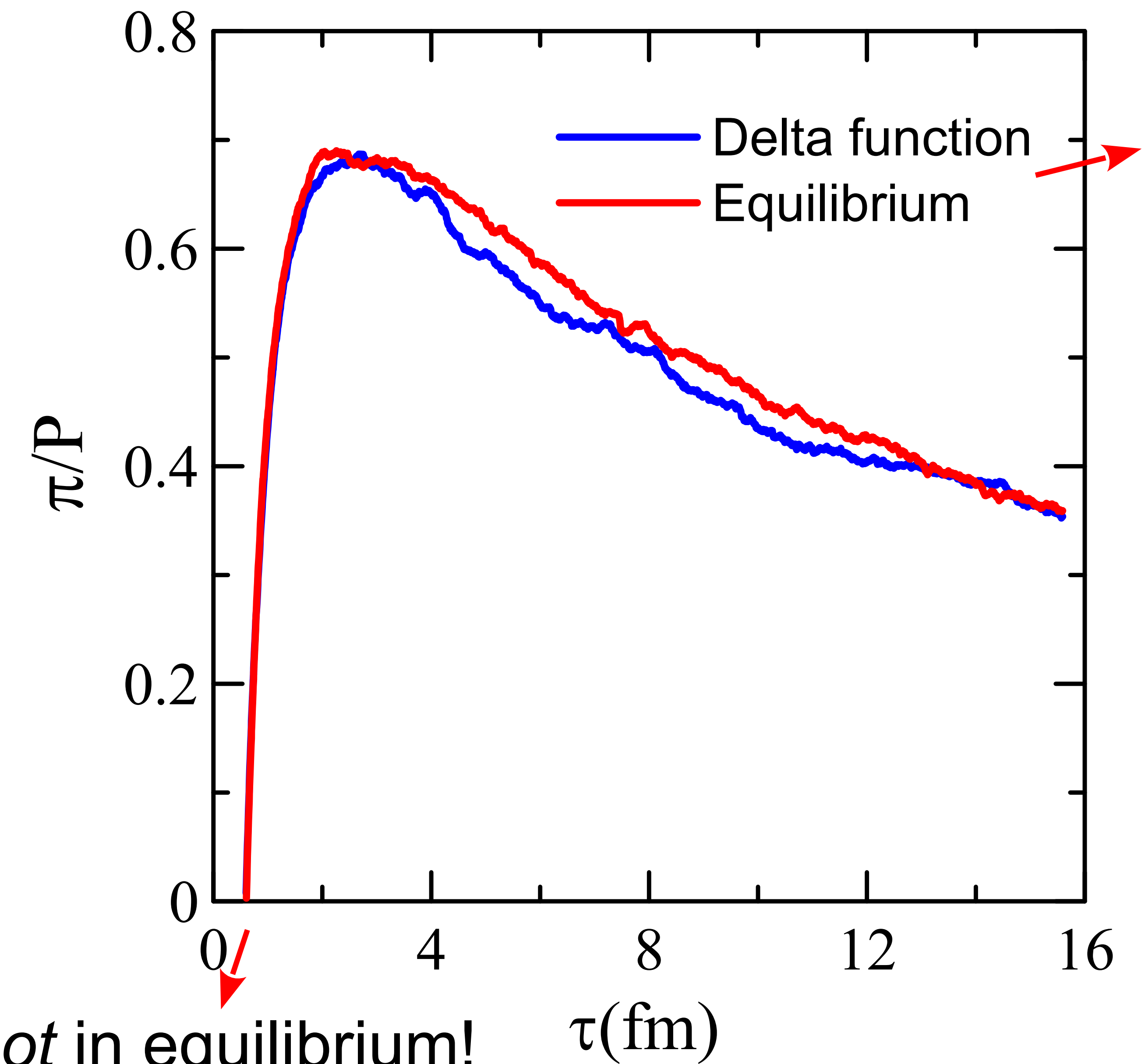
$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} - \frac{2I_{40}}{3I_{50}}\sigma\pi^{\lambda\langle\mu}\pi^{\nu\rangle}_{\lambda} = 2\frac{\eta}{\tau_\pi}\sigma^{\mu\nu} - 2\sigma^{\langle\mu}_{\lambda}\pi^{\nu\rangle\lambda} - \frac{4}{3}\pi^{\mu\nu}\theta$$

$f_{0\mathbf{k}}$	$\lambda\exp\left(-E_{\mathbf{k}}/\Lambda\right)$	$\lambda\Theta\left(\Lambda - E_{\mathbf{k}}\right)$	$\lambda\delta\left(E_{\mathbf{k}} - \Lambda\right)$
τ_π	$\frac{9}{5}\ell_{\mathrm{mfp}}$	$\frac{18}{13}\ell_{\mathrm{mfp}}$	$\frac{9}{7}\ell_{\mathrm{mfp}}$
η	$\frac{6}{5}\frac{T}{\sigma_T}$	$\frac{84}{65}\frac{T}{\sigma_T}$	$\frac{9}{7}\frac{T}{\sigma_T}$

Coefficients do not change much with f_{0k}. Can we see this?

Simulation: Boltzmann eq. + Bjorken flow

ultrarelativistic classical gas of *hard spheres*



Initial conditions, fixed energy

shear viscosity

$$\frac{\eta}{n} \approx 6$$

Evolution of shear stress does not see this non-equilibrium effect

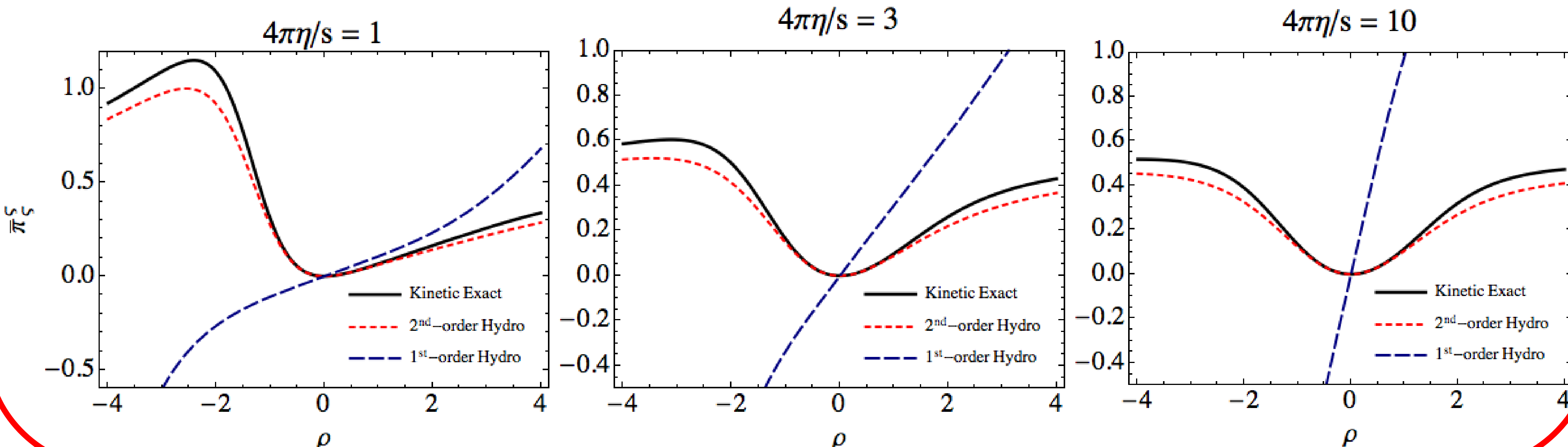
not in equilibrium!

Is it ok to treat anisotropic part as a correction?

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + f_{0\mathbf{k}}\left(1 - a f_{0\mathbf{k}}\right)\phi_{\mathbf{k}}$$

isotropic — *Anisotropic corrections*

This appears to be a good approximation so far ...
(checked in Gubser or Bjorken flow)



Otherwise, one gets anisotropic hydrodynasmics

Conclusions

- The applicability of fluid-dynamical models of heavy ion collisions cannot be easily justified

- The derivation of hydrodynamics using the method of moments is more general than previously considered: **hydrodynamic equations can be obtained even expanding around far from equilibrium.**