Dynamical systems and nonlinear transient rheology of the far-from-equilibrium Bjorken flow

Syo Kamata
Jiangxi Normal U / NC State U

Collaborators:
A. Behtash, M. Martinez, H. Shi (NC State U)
C. N. Cruz-Camacho (U Nacional de Colombia)

A. Behtash, SK, M. Maritinez, C.N. Cruz, Phys. Lett. B 797 (2019) 134914
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New development of hydrodynamics and its applications in Heavy-Ion Collisions

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Motivation

- Hydro works fairly well in Pb-Pb, pA and p-p collisions. [Willer & Rometschke,Werner et. al., Bozek]
- Different theoretical models indicate hydro works even when the pressure anisotropy.

  → Hydro dynamics can be valid in far-from-equilibrium situations.

  [Heller, Spalinski, Strickland, Martinez, Ryblewski, Florkowski, Romatschke, Casalderrey, Noronha, Denicol. Jaiswal, ...]
- The onset of hydrodynamics is determined by the decay of nonhydro modes.

Our problems and ideas

- Asymptotic behaviors including nonhydro modes in the far-from-equilibrium
  - Distribution function
  - Energy density, pressure, shear tensor,...
  - Transport coefficients

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- Asymptotic behaviors including nonhydro modes in the far-from-equilibrium
  - Distribution function
  - Energy density, pressure, shear tensor,...
  - Transport coefficients

- Analysis based on dynamical system
  - (Nonautonomous) Morse decomposition
  - Transseries analysis (Resurgence)

Overview

Boltzmann equation (Bjorken flow)

$$\partial_\tau f\left(\tau, p_T, p_\varsigma\right) = -\frac{1}{\tau_r(\tau)}\left[f\left(\tau, p_T, p_\varsigma\right) - f_{eq.}\left(-u\cdot p/T\right)\right],$$

Chapman-Enskog expansion

Hydrodynamics

EM tensor
Transport coeffs

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Chapman-Enskog expansion

Mode expansion

Hydrodynamics

EM tensor
Transport coeffs

Late time limit

Dynamical system
(Non-autonomous system)

Transseries Analysis
(Resurgence)

Bjorken flow (+ RTA approximation)

[Bjorken]

- Isometry: $ISO(2) \times SO(1,1) \times \mathbb{Z}_2$
  - Depends on $\tau := \sqrt{t^2 - z^2},\ p_T := \sqrt{p_x^2 + p_y^2},\ p_\zeta$
- Scale invariant (Traceless EM tensor) $T^\mu_{\ \mu} = 0$
  - Massless particle, vanishing bulk viscosity
- RTA approximation

$$Q[f,f] \ \rightarrow \ -\frac{T^2}{C_\mathbf{p}}(f - f_{\rm eq}), \quad C_\mathbf{p} := -T^2\tau_r/(u\cdot p) \text{ with } \tau_r = \theta_0/T \qquad \theta_0 = 5\eta_0/s$$

- Maxwell-Boltzmann equilibrium $f_{eq}(\tau, p) = \exp\left(-\frac{p\cdot u}{T(\tau)}\right)$

$$\Rightarrow \qquad \frac{\partial f}{\partial \tau} = -\frac{T}{\theta_0}\left(f - f_{eq}\right)$$

Setup

- Milne coordinate: $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -\tau^2),$
  - $\tau := \sqrt{t^2 - z^2},\ \tanh\zeta := z/t$
- Local rest frame $u^\mu = (1, 0, 0, 0)$
  - Landau frame $T^{\mu\nu} u_\nu = \lambda u^\mu$
  - Vanishing heat flow

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  - Vanishing heat flow
- Ansatz: Mode expansion **[Grad, Romatschke et. al, ...]**

$$f(\tau, p) := \sum_{n,\ell=0} c_{n\ell}(\tau) \mathcal{P}_{2\ell}(p_\zeta/(T\tau)) \mathcal{L}_n^{(3)}(p^\tau/T) f_{eq}(p^\tau/T)$$

$$f_{eq}(x) := \exp(-x) \qquad p^\tau := \sqrt{p_T^2 + (p_\zeta/\tau)^2}$$

$\mathcal{P}_\ell(x)$: Legendre polynomial $\quad \mathcal{L}_n^{(\alpha)}(x)$: Laguerre polynomial

Hydro variables

- Observables: EM tensor

$$T^{\mu\nu} := \langle p^\mu p^\nu \rangle_f = \mathrm{diag}(\mathcal{E}, P_T, P_T, P_L/\tau^2)$$

$$\mathcal{E} = \frac{3T^4}{\pi^2} c_{00},\ P_T = \frac{\mathcal{E}}{3}\left(1 - \frac{1}{5}c_{01}\right),\ P_L = \frac{\mathcal{E}}{3}\left(1 + \frac{2}{5}c_{01}\right) \qquad P_T, P_L \in \mathbb{R}_0^+$$

$$\langle \mathcal{O} \rangle_f (\tau) := \int_p \mathcal{O}(\tau, p) f(\tau, p),\ \int_p := \frac{d^2 p_T dp_\zeta}{(2\pi)^3 \tau p^\tau}$$

Nontrivial viscous component: $\pi^{\zeta\zeta} := \frac{\mathcal{E}}{\tau^2}\bar{\pi},\ \bar{\pi} = \frac{2}{3}\left(\frac{P_L - P_T}{\mathcal{E}}\right) = \frac{2}{15}c_{01}$

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- Assume the energy conservation law

$$D_\mu T^{\mu\nu} = 0$$

Chapman-Enskog expansion

[Chapman, Enskog, Grad,..]

- Introduce a book-keeping parameter
  - Ansatz for f with $\alpha \in \mathbb{R}^+$

$$f = f_{eq} + \alpha f_{(1)} + \alpha^2 f_{(2)} + \cdots \qquad Q[f,f] \;\to\; \frac{1}{\alpha} Q[f,f]$$

  - Assumption:

$$|f_{eq}| \gg |f_{(1)}| \gg |f_{(2)}| \gg \cdots \quad \text{as} \quad \mathrm{Kn} \ll 1$$

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$$|f_{eq}| \gg |f_{(1)}| \gg |f_{(2)}| \gg \cdots \quad \text{as} \quad \mathrm{Kn} \ll 1$$

$$\begin{aligned} c_{01} &= -\frac{8\tau_r}{5\pi^2\tau} - \frac{64\tau_r^2}{105\pi^2\tau^2} + \mathcal{O}(1/\tau^3) \\ &= -\frac{2\eta}{\tau T^4} + \frac{4}{3\tau^2 T^4}(\lambda_1 - \eta\tau_\pi) + \mathcal{O}(1/\tau^3), \end{aligned} \qquad \begin{aligned} c_{02} &= \frac{32\tau_r^2}{21\pi^2\tau^2} + \mathcal{O}(1/\tau^3) \\ &= \frac{4}{3\tau^2 T^4}(\lambda_1 + \eta\tau_\pi) + \mathcal{O}(1/\tau^3). \end{aligned}$$

$$\mathrm{Kn} \sim w^{-1} := 1/(T(\tau)\tau)$$

Derivation of ODEs

- Projection onto $c_{n\ell}$

$$c_{n\ell}(\tau) = \frac{2\pi^2(4\ell+1)}{T^4}\frac{\Gamma(n+1)}{\Gamma(n+4)}\left\langle (p^\tau)^2 \mathcal{P}_{2\ell}(p_\zeta/(\tau p^\tau))\mathcal{L}_n^{(3)}(p^\tau/T)\right\rangle_f$$

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- ODEs from Boltzmann eq.
  - Simultaneous, nonlinear, and nonautonomous

$$\frac{dc_{n\ell}}{dw} + \frac{1}{1-\frac{c_{01}}{20}}\left[\frac{3}{2w}\{\alpha_{n\ell}c_{n\ell+1} + \beta_{n\ell}c_{n\ell} + \gamma_{n\ell}c_{n\ell-1} - n\left(\rho_\ell c_{n-1\ell+1} + \psi_\ell c_{n-1\ell} + \phi_\ell c_{n-1\ell-1}\right)\}\right] + \frac{3}{2\theta_0}\left(c_{n\ell} - \delta_{n,0}\delta_{\ell,0}\right) = 0$$

$$w := T(\tau)\tau$$

$$c_{00} = 1$$

$$\alpha_{n\ell} := \frac{(2\ell+2)(2\ell+1)(n-2\ell+1)}{(4\ell+3)(4\ell+5)},\ \beta_{n\ell} := \frac{2\ell(2\ell+1)(2n+5)}{3(4\ell-1)(4\ell+3)} - \frac{(n+4)}{30}c_{01},\ \gamma_{n\ell} := \frac{2\ell(2\ell-1)(n+2\ell+2)}{(4\ell-3)(4\ell-1)},$$

$$\rho_\ell := \frac{(2\ell+1)(2\ell+2)}{(4\ell+3)(4\ell+5)},\ \psi_\ell := \frac{4\ell(2\ell+1)}{3(4\ell-1)(4\ell+3)} - \frac{c_{01}}{30},\ \phi_\ell := \frac{2\ell(2\ell-1)}{(4\ell-3)(4\ell-1)},$$

Derivation of ODEs

- Projection onto $c_{n\ell}$

$$c_{n\ell}(\tau) = \frac{2\pi^2(4\ell+1)}{T^4}\frac{\Gamma(n+1)}{\Gamma(n+4)}\left\langle (p^\tau)^2 \mathcal{P}_{2\ell}(p_\zeta/(\tau p^\tau))\mathcal{L}_n^{(3)}(p^\tau/T)\right\rangle_f$$

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$$w := T(\tau)\tau$$

$$c_{00} = 1$$

➡ We take $L := \max(\ell), N := \max(n) = 0,$
but $c_{0\ell}$ can be determined only by the $n=0$ sector
for a generic N.

Global structure (w-variable)

$$\frac{d\mathbf{c}}{dw} = \mathbf{f}(w,\mathbf{c}), \qquad \mathbf{f}(w,\mathbf{c}) = -\frac{1}{1-\frac{c_{01}}{20}}\left[\hat{\Lambda}\mathbf{c} + \frac{1}{w}(\mathfrak{B}\mathbf{c} + c_{01}\mathfrak{D}\mathbf{c} + \mathbf{A})\right],$$

$$\mathbf{c} := (c_{01}, c_{02}, \cdots, c_{0L})^{\top}$$

- Definition of fixed points

$$\lim_{w\to 0_+} \mathbf{f}(w,\mathbf{c}^*) = 0 \ \text{ for UV}, \qquad \lim_{w\to +\infty} \mathbf{f}(w,\mathbf{c}^*) = 0 \ \text{ for IR}.$$

  - Two UV (early time) fixed points

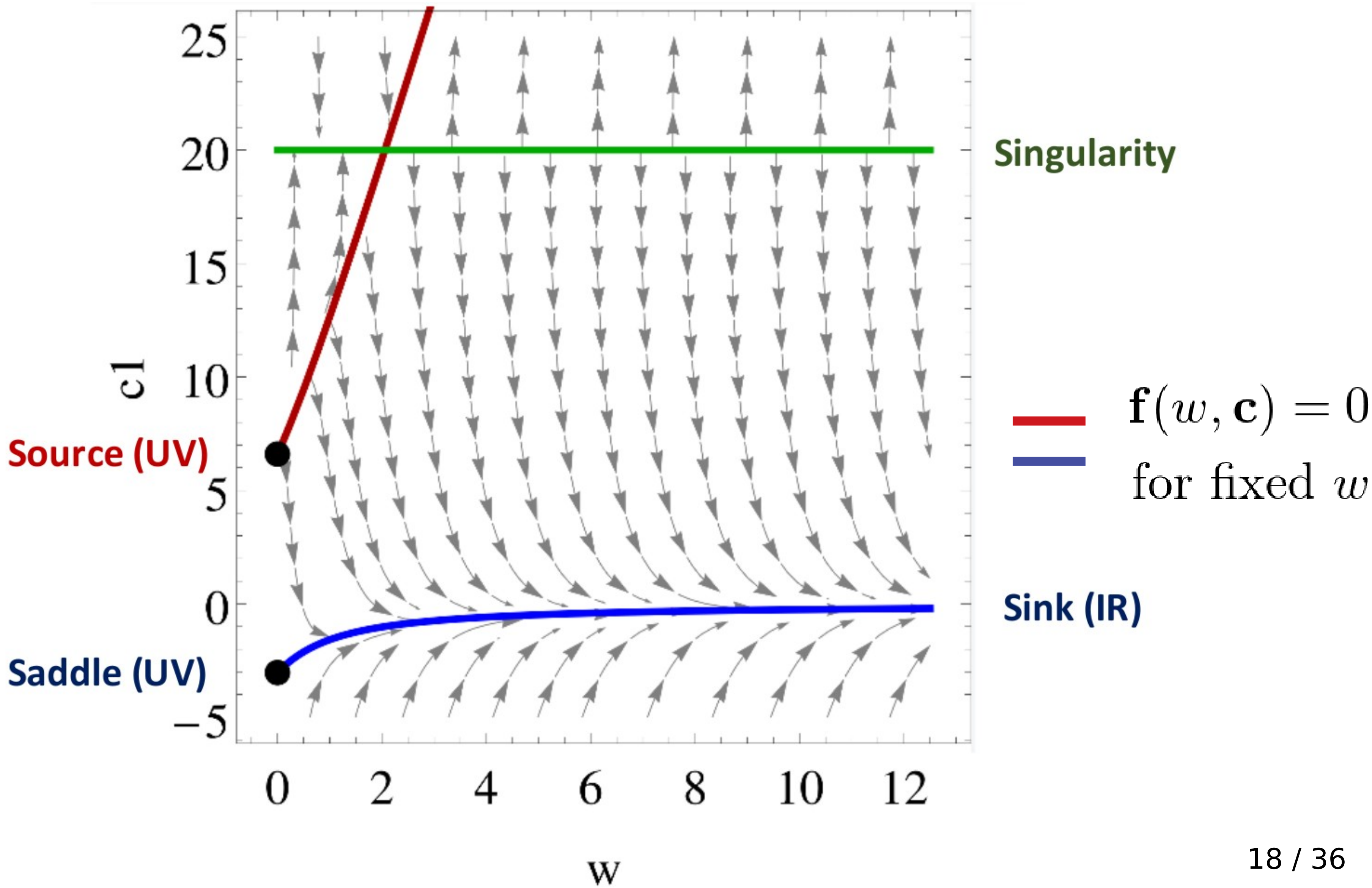
$$\mathbf{c}^*_+ = (5, \{4\ell+1\}_{\ell=2}^{\infty}) \qquad \mathbf{c}^*_- = \left(-5/2, \left\{(-4)^{-\ell}(4\ell+1)\binom{2\ell}{\ell}\right\}_{\ell=2}^{\infty}\right)$$

  - One IR (late time) fixed point

$$\mathbf{c}^* = (0, \cdots, 0)$$

- Singularity at $c_{01} = 20$

Global structure (L=1, N=0)



Transseries anlysis

- Formal transseries
  - Extension of an asymptotic power expansion

$$A(x) \sim \sum_{k=0} A_k x^{-k} \quad \text{as } x \to +\infty$$

$$\Rightarrow \quad A(x) \sim \sum_{n,k=0} A_{nk} \; e^{-nbx} x^{-n\beta-k} \quad \text{as } x \to +\infty$$

  - Polynomial ring (function)
  - Coeffs and basis can be determined by ODEs

- Divergent series and Borel-nonsummable

Formal transseries solution

[Costin]

$$\frac{d\mathbf{c}}{dw} = \mathbf{f}(w, \mathbf{c}), \qquad \mathbf{f}(w, \mathbf{c}) = -\frac{1}{1 - \frac{c_{01}}{20}} \left[\hat{\Lambda}\mathbf{c} + \frac{1}{w}(\mathfrak{B}\mathbf{c} + c_{01}\mathfrak{D}\mathbf{c} + \mathbf{A}) \right],$$

- Transseries ansatz $\tilde{\mathbf{c}}(w) = U\mathbf{c}(w), \quad U \in \mathbb{C}^{I\times I} \qquad I := \dim(\mathbf{c}),$

$$\tilde{c}_i(w) = \sum_{|\mathbf{m}|\geq 0}^{\infty} \sum_{k=0}^{\infty} \tilde{u}_{i,k}^{(\mathbf{m})} E_k^{(\mathbf{m})}(w) \qquad \in \mathbb{C}[[w^{-1}, \zeta_1, \cdots, \zeta_I]] \qquad \mathbf{m} \in \mathbb{N}_0^I,$$

$$E_k^{(\mathbf{m})}(w) := \boldsymbol{\zeta}^{\mathbf{m}}(w) w^{-k}, \qquad \boldsymbol{\zeta}^{\mathbf{m}} := [\zeta_i(w)]^{m_i}, \qquad \zeta_i(w) := \sigma_i e^{-S_i w} w^{\tilde{b}_i},$$

Integration constants: $\sigma_i \in \mathbb{C},$ $\qquad u_{j,0}^{(\mathbf{m})} = \delta_{ij} \text{ for } m_j = \delta_{ij}.$

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Integration constants: $\sigma_i \in \mathbb{C}$, $\qquad u_{j,0}^{(\mathbf{m})} = \delta_{ij}$ for $m_j = \delta_{ij}$.

- Transseries from ODEs

$$20\left[(\mathbf{m}\cdot\tilde{\mathbf{b}} + b_i - k)\tilde{u}_{i,k}^{(\mathbf{m})} + \left(\frac{3}{2\theta_0} - \mathbf{m}\cdot\mathbf{S}\right)\tilde{u}_{i,k+1}^{(\mathbf{m})}\right]$$
$$+20\tilde{A}_i\delta_{k,0}\delta_{\mathbf{m},\mathbf{0}} - \sum_{|\mathbf{m}'|\geq 0}^{\mathbf{m}}\left[\sum_{k'=0}^{k}(\mathbf{m}'\cdot\tilde{\mathbf{b}} + 4 - k')u_{1,k-k'}^{(\mathbf{m}-\mathbf{m}')}\tilde{u}_{i,k'}^{(\mathbf{m}')}\right.$$
$$\left.-\mathbf{m}'\cdot\mathbf{S}\sum_{k'=0}^{k+1}u_{1,k-k'+1}^{(\mathbf{m}-\mathbf{m}')}\tilde{u}_{i,k'}^{(\mathbf{m}')}\right] = 0,$$

$$\hat{\mathfrak{B}} = U\mathfrak{B}U^{-1} = \operatorname{diag}(b_1,\ldots,b_L) \in \mathbb{C}^L,$$
$$\tilde{\mathbf{A}} = U\mathbf{A},$$
$$\hat{\Lambda} = U\hat{\Lambda}U^{-1}.$$

$$\Rightarrow \quad S_i = \frac{3}{2\theta_0}, \qquad \tilde{b}_i = -\left(b_i - \frac{1}{5}\right).$$

Trans-asymptotic matching

[Basar et. al., ...]

- Renormalized transport coefficients

$$c_i(w) = \sum_{i=1}^{I} U^{-1}_{ij}\tilde{c}_j(w), \quad \tilde{c}_i(\zeta(w)) = \sum_{k=0} \tilde{C}_{i,k}(\zeta(w))w^{-k}, \quad \tilde{C}_{i,k}(\zeta(w)) = \sum_{|\mathbf{m}|\geq 0} \tilde{u}^{(\mathbf{m})}_{i,k}\zeta^{\mathbf{m}}|_{\zeta_i=\zeta_i(w)}$$

$$c_1^{1\mathrm{st}}(w\to\infty) = -\frac{8}{3}\frac{\theta_0}{w} = -\frac{40}{3}\frac{(\eta/s)_0}{w},$$

$$c_1^{2\mathrm{nd}}(w\to\infty) = -\frac{32}{63}\frac{\theta_0^2}{w^2} = -\frac{40}{9}T\left[\tau_\pi\left(\frac{\eta}{s}\right)_0 - \left(\frac{\lambda_1}{s}\right)_0\right]\frac{1}{w^2},$$

$$\Rightarrow$$

$$\left(\frac{\eta}{s}\right)_{\mathrm{reno}} = -\frac{3}{40}C_{1,1}(w),$$

$$T\left[\tau_\pi\frac{\eta}{s} - \frac{\lambda_1}{s}\right]_{\mathrm{reno}} = \frac{9}{40}C_{1,2}(w),$$

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$$T\left[\tau_\pi \frac{\eta}{s} - \frac{\lambda_1}{s}\right]_{\mathrm{reno}} = \frac{9}{40} C_{1,2}(w),$$

- PDE for a fixed k

$$20\left[((\tilde{\mathbf{b}} \cdot \hat{\boldsymbol{\xi}} - k) + b_i)\tilde{C}_{i,k} - \mathbf{S} \cdot \hat{\boldsymbol{\xi}} \tilde{C}_{i,k+1} + \frac{3}{2\theta_0}\tilde{C}_{i,k+1}\right]$$

$$+ 20\tilde{A}_i \delta_{k,0} - \sum_{k'=0}^{k} C_{1,k-k'}(\tilde{\mathbf{b}} \cdot \hat{\boldsymbol{\xi}} - k')\tilde{C}_{i,k'}$$

$$+ 20 \sum_{i'=1}^{I}\sum_{k'=0}^{k} C_{1,k-k'} \tilde{\mathfrak{D}}_{ii'} \tilde{C}_{i',k'} + \sum_{k'=0}^{k+1} C_{1,k-k'+1} \mathbf{S} \cdot \hat{\boldsymbol{\xi}} \tilde{C}_{i,k'} = 0,$$

$$\hat{\zeta}_i := \partial/\partial \log \zeta_i,$$

Initial conditions: $\tilde{u}_{i,k}^{(\mathbf{0})}$

Trans-asymptotic matching

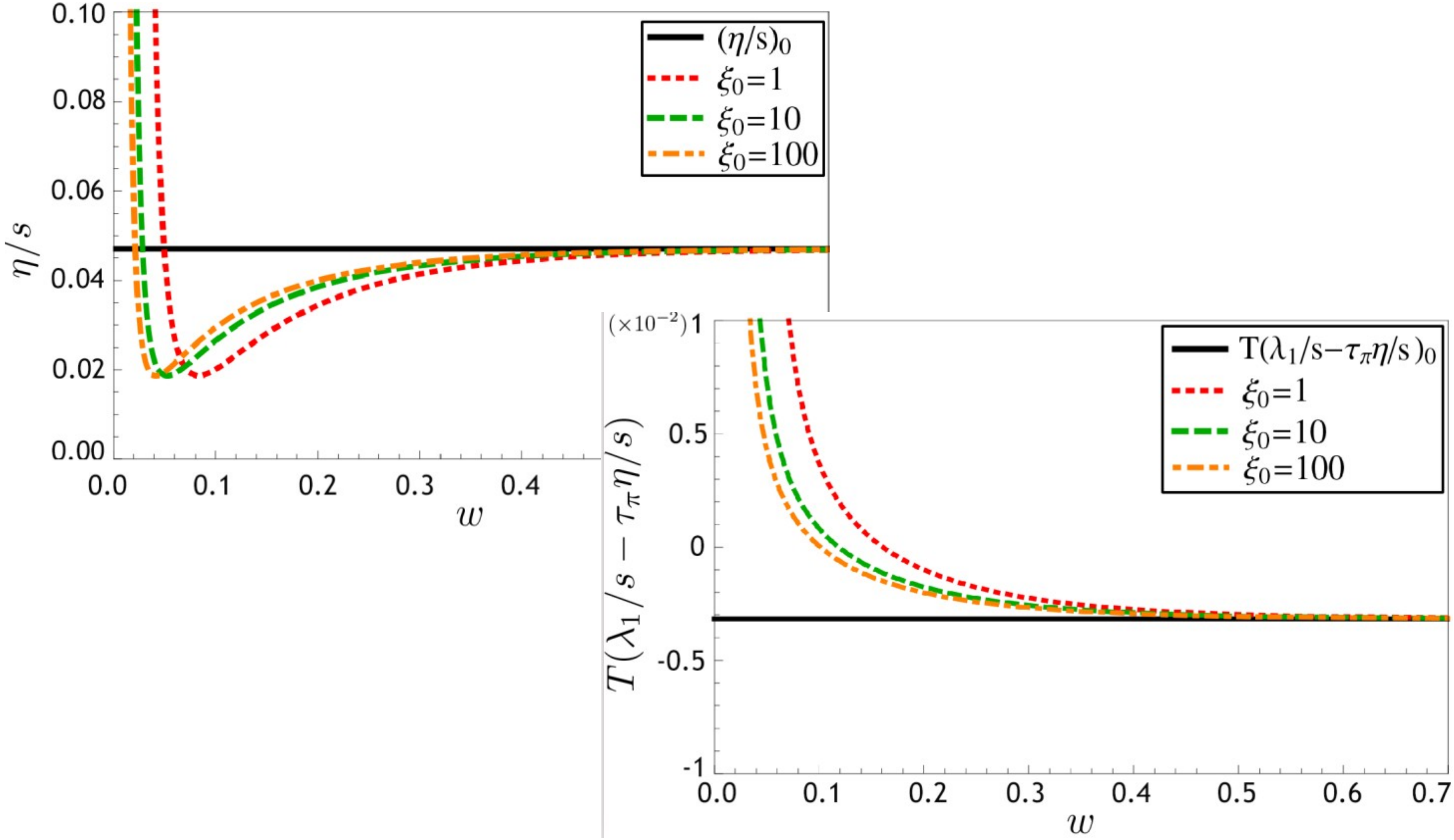
- Exact solutions for L=1 and N=0

$$C_{1,0}(\zeta) = -20W_\zeta, \qquad C_{1,1}(\zeta) = -\frac{8\theta_0(50W_\zeta^3 + 105W_\zeta^2 + 36W_\zeta + 5)}{15(W_\zeta + 1)},$$

$$C_{1,2}(\zeta) = -\frac{8\theta_0^2}{7875(W_\zeta + 1)}\left[\frac{25(700W_\zeta^4 + 2195W_\zeta^3 + 966W_\zeta^2 + 20)}{W_\zeta} + \frac{4032}{(W_\zeta + 1)^2} + 3685\right],$$

$$W_\zeta := W(-20\zeta), \quad W(x)\text{: Lambert } W \text{ function}$$

Trans-asymptotic matching



Conclusion

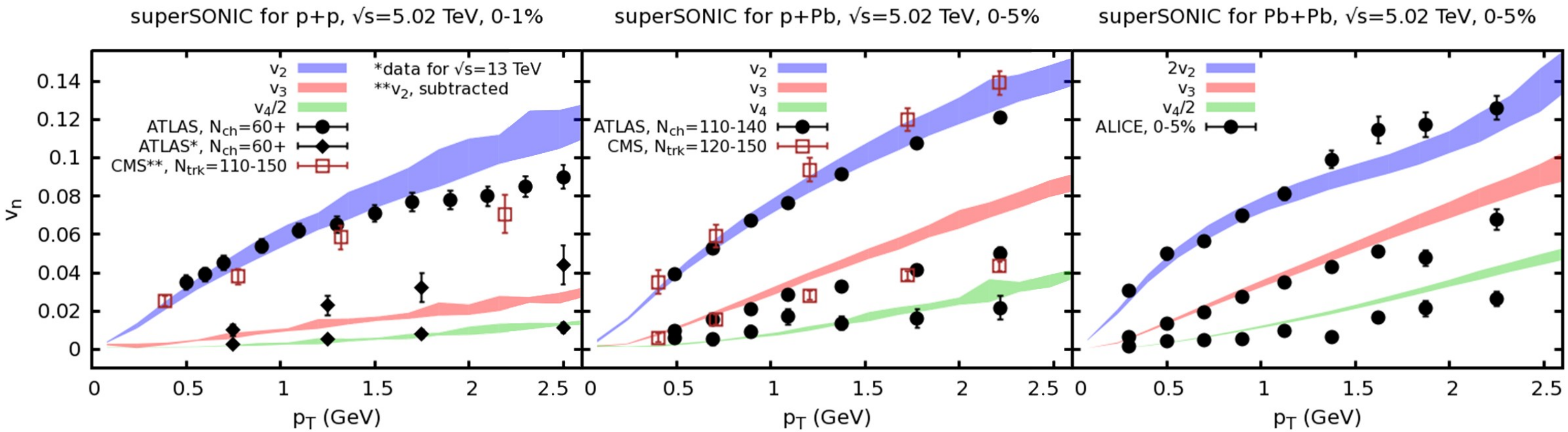
- We proposed the transseries analysis based on dynamical system approach for the RTA Bjorken flow.
- Asymptotic behavior of distribution and observables including nonhydro modes can be determined.
- Renormalized transport coefficients with the early time history is available by summation of nonhydro series.

Outlook

- Gubser flow
- External field
- More realistic collision of kernel
- Non-relativistic system (condensed matter)
- Generic hydro (e.g. vortex fluid)
  - ODE $\rightarrow$ PDE
- Etc..

Back-up slides

Motivation



- Hydro works fairly well in Pb-Pb, pA and p-p collisions [Willer & Rometschke,Werner et. al., Bozek]

Relativistic kinetic theory

- Physics of far-from-equilibrium

$$\frac{d^6 N(t,\mathbf{x},\mathbf{p})}{d^3\mathbf{x} d^3\mathbf{p}} = f(t,\mathbf{x},\mathbf{p}) \in \mathbb{R}_0^+, \quad p^\mu \partial_\mu f(t,\mathbf{x},\mathbf{p}) = Q[f,f].$$

- Collision of Kernel $Q[f,f]$

$$Q[f,f] := \int_{\mathbf{p}_2} \int d\Omega \sigma(\Omega) |\mathbf{p}_1 - \mathbf{p}_2| \left[f(\mathbf{p}_2{}') f(\mathbf{p}_1{}') - f(\mathbf{p}_2) f(\mathbf{p}_1) \right]$$

  - Violation of the detail valance
  - H-function $H(t) := \int_{\mathbf{p}} f(t,\mathbf{p}) \log f(t,\mathbf{p}), \quad \frac{dH(t)}{dt} \le 0.$

$$Q[f,f] = 0 \Leftrightarrow \frac{dH}{dt} = 0$$

Kinetic theory → Hydro

A. N. Gorban I. Karlin arXiv:1310.0406 [math-ph]

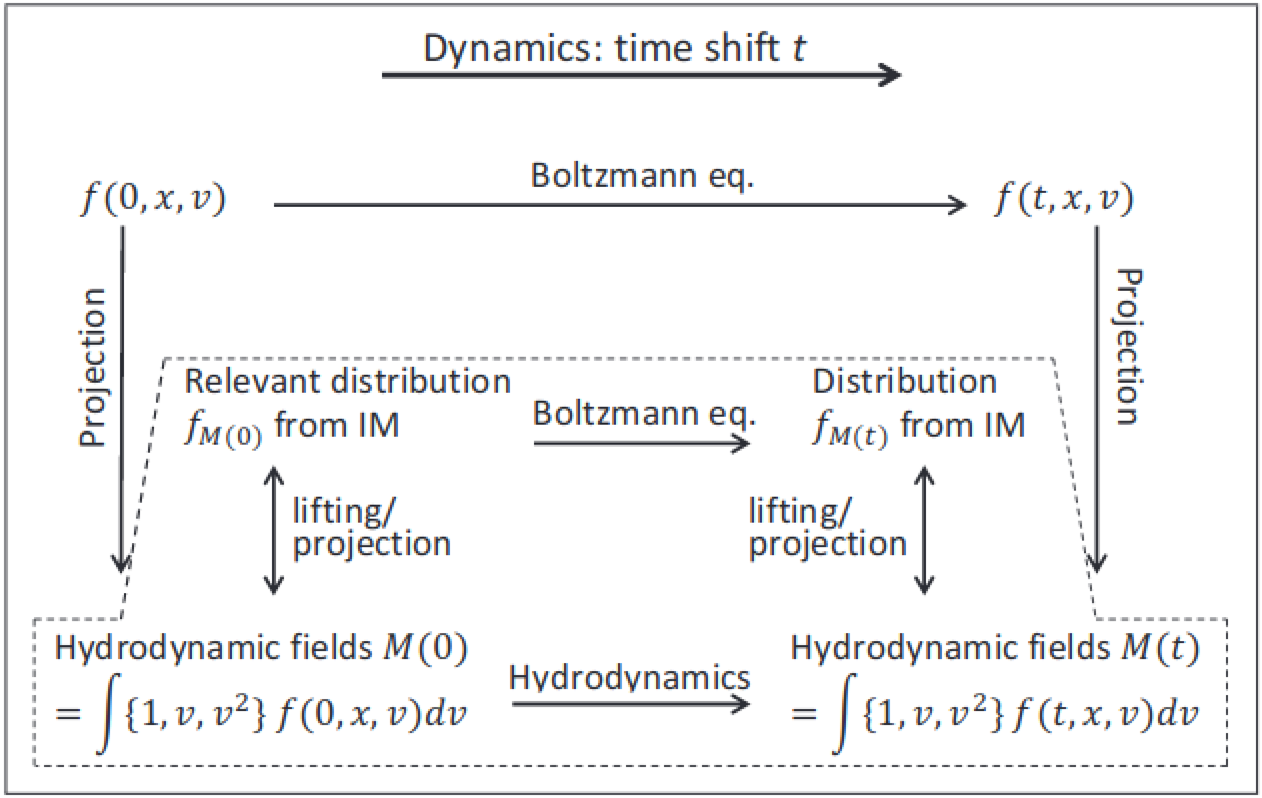


FIGURE 2. *McKean diagram.* The Chapman–Enskog procedure aims to create a lifting operation, from the hydrodynamic variables to the corresponding distributions on the invariant manifold. IM stands for Invariant Manifold. The part of the diagram in the dashed polygon is commutative.

Fast-slow decomposition

A. N. Gorban I. Karlin arXiv:1310.0406 [math-ph]

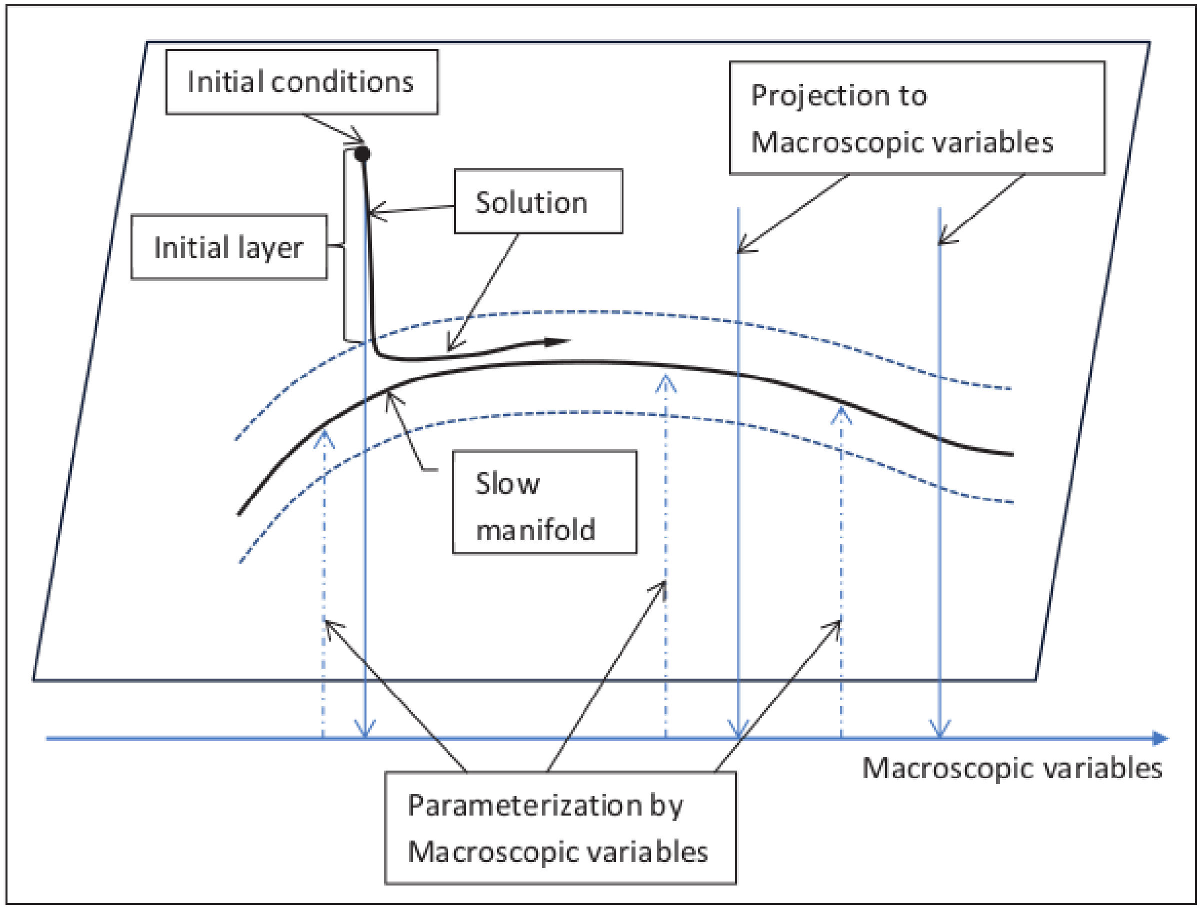


Figure 1. Fast–slow decomposition. Bold dashed lines outline the vicinity of the slow manifold where the solutions stay after initial layer. The projection of the distributions onto the hydrodynamic fields and the parametrization of this manifold by the hydrodynamic fields are represented.

Chapman-Enskog expansion

- Introduce a book-keeping parameter
  - Expand f by $\alpha \in \mathbb{R}^+$

$$f = f_{eq} + \alpha f_{(1)} + \alpha^2 f_{(2)} + \cdots \qquad\qquad Q[f,f] \;\rightarrow\; \frac{1}{\alpha} Q[f,f]$$

  - Assumption:

$$|f_{eq}| \gg |f_{(1)}| \gg |f_{(2)}| \gg \cdots \quad \text{as} \quad \mathrm{Kn} \ll 1$$

$$\begin{aligned} f - f_{eq} = \chi_p C_p \Big[- \tilde{p} \frac{20}{9T^2\tau^2} - \tilde{p}^2 \frac{4}{9T^2\tau^2} - \tilde{p}^3 \frac{4}{9T^2\tau^2} - \tilde{p}^5 \frac{64}{315T^2\tau^2} + \tilde{p}^6 \frac{16}{315T^2\tau^2} + \cdots \Big] P_0(\cos\theta) \\ + \Big[-\tilde{\chi}_p \tilde{p}^2 \left(\frac{2}{3\tau T}\right) + \tilde{\chi}'_p \tilde{C}_p \tilde{p}^4 \left(\frac{8}{63\tau^2 T^2}\right) - \tilde{\chi}_p \tilde{C}_p \tilde{p}^3 \left(\frac{8}{9\tau^2 T^2}\right) + \cdots \Big] P_2(\cos\theta) \\ + \Big[\tilde{\chi}'_p \tilde{C}_p \tilde{p}^4 \left(\frac{8}{35\tau^2 T^2}\right) + \cdots \Big] P_4(\cos\theta), \end{aligned}$$

$$\chi_{\mathbf{p}} := -\frac{C_{\mathbf{p}}}{2} f'_{eq}, \qquad \tilde{\chi}_{\mathbf{p}} := -C_{\mathbf{p}} f'_{eq}, \qquad \tilde{p} := p/T,$$

Analysis of Bjorken flow

- Projection onto $c_{n\ell}$

$$c_{n\ell}(\tau) = \frac{2\pi^2(4\ell+1)}{T^4}\frac{\Gamma(n+1)}{\Gamma(n+4)}\left\langle (p^\tau)^2\mathcal{P}_{2\ell}(p_\zeta/(\tau p^\tau))\mathcal{L}_n^{(3)}(p^\tau/T)\right\rangle_f$$

- ODEs for modes from Boltzmann eq.
  - Simultaneous, nonlinear, and nonautonomous

$$\frac{dc_{n\ell}}{d\tau} + \frac{1}{\tau}\left[\alpha_{n\ell}c_{n\ell+1} + \beta_{n\ell}c_{n\ell} + \gamma_{n\ell}c_{n\ell-1} - n\left(\rho_\ell c_{n-1\ell+1} + \psi_\ell c_{n-1\ell} + \phi_\ell c_{n-1\ell-1}\right)\right] + \frac{T}{\theta_0}\left(c_{n\ell} - \delta_{n,0}\delta_{\ell,0}\right) = 0$$

From conservation law

$$\frac{dT}{d\tau} = -\frac{T}{3\tau}\left(1 + \frac{c_{01}}{10}\right)$$

$$c_{00} = 1$$

$$\alpha_{n\ell} := \frac{(2\ell+2)(2\ell+1)(n-2\ell+1)}{(4\ell+3)(4\ell+5)},\ \beta_{n\ell} := \frac{2\ell(2\ell+1)(2n+5)}{3(4\ell-1)(4\ell+3)} - \frac{(n+4)}{30}c_{01},\ \gamma_{n\ell} := \frac{2\ell(2\ell-1)(n+2\ell+2)}{(4\ell-3)(4\ell-1)},$$

$$\rho_\ell := \frac{(2\ell+1)(2\ell+2)}{(4\ell+3)(4\ell+5)},\ \psi_\ell := \frac{4\ell(2\ell+1)}{3(4\ell-1)(4\ell+3)} - \frac{c_{01}}{30},\ \phi_\ell := \frac{2\ell(2\ell-1)}{(4\ell-3)(4\ell-1)},$$

Analysis of Bjorken flow

- Vectorial form of the ODEs

$$\frac{d\mathbf{c}}{dw} = \mathbf{f}(w,\mathbf{c}), \qquad \mathbf{f}(w,\mathbf{c}) = -\frac{1}{1-\frac{c_{01}}{20}}\left[\hat{\Lambda}\mathbf{c} + \frac{1}{w}(\mathfrak{B}\mathbf{c} + c_{01}\mathfrak{D}\mathbf{c} + \mathbf{A})\right],$$

$$\mathbf{c} = (\underbrace{c_{01},\ldots,c_{0L}}_{L}, \underbrace{c_{10},c_{11},\ldots,c_{1L}}_{L+1},\ldots,c_{N0},\ldots,c_{NL})^\top, \qquad \hat{\Lambda} = \mathrm{diag}\left(\frac{3}{2\theta_0},\ldots,\frac{3}{2\theta_0}\right),$$

$$\mathbf{A} = \frac{3}{2}(\underbrace{\gamma_{01},0,\ldots,0}_{L},0,\phi_1,0,\ldots,0)^\top,$$

$$\mathfrak{B} = \frac{3}{2}\begin{pmatrix} \bar{\mathfrak{B}}_{00} & & & & \\ \bar{\mathfrak{B}}_{10} & \bar{\mathfrak{B}}_{11} & & & \\ & \bar{\mathfrak{B}}_{21} & \bar{\mathfrak{B}}_{22} & & \\ & & \ddots & \ddots & \\ & & & \bar{\mathfrak{B}}_{NN-1} & \bar{\mathfrak{B}}_{NN} \end{pmatrix}, \qquad \mathfrak{D} = \begin{pmatrix} \bar{\mathfrak{D}}_{00} & & & & \\ \bar{\mathfrak{D}}_{10} & \bar{\mathfrak{D}}_{11} & & & \\ & \bar{\mathfrak{D}}_{21} & \bar{\mathfrak{D}}_{22} & & \\ & & \ddots & \ddots & \\ & & & \bar{\mathfrak{D}}_{NN-1} & \bar{\mathfrak{D}}_{NN} \end{pmatrix},$$

Analysis of Bjorken flow

$$\mathfrak{B} = \frac{3}{2}\begin{pmatrix} \bar{\mathfrak{B}}_{00} & & & & & \\ \bar{\mathfrak{B}}_{10} & \bar{\mathfrak{B}}_{11} & & & & \\ & \bar{\mathfrak{B}}_{21} & \bar{\mathfrak{B}}_{22} & & & \\ & & \ddots & \ddots & & \\ & & & & \bar{\mathfrak{B}}_{NN-1} & \bar{\mathfrak{B}}_{NN} \end{pmatrix}, \qquad \mathfrak{D} = \begin{pmatrix} \bar{\mathfrak{D}}_{00} & & & & & \\ \bar{\mathfrak{D}}_{10} & \bar{\mathfrak{D}}_{11} & & & & \\ & \bar{\mathfrak{D}}_{21} & \bar{\mathfrak{D}}_{22} & & & \\ & & \ddots & \ddots & & \\ & & & & \bar{\mathfrak{D}}_{NN-1} & \bar{\mathfrak{D}}_{NN} \end{pmatrix},$$

$$\bar{\mathfrak{B}}_{00} = \begin{pmatrix} \frac{2}{3}\Omega_{01} & \alpha_{01} & & & \\ \gamma_{02} & \frac{2}{3}\Omega_{02} & \alpha_{02} & & \\ & \ddots & \ddots & \ddots & \\ & & \gamma_{0L-1} & \frac{2}{3}\Omega_{0L-1} & \alpha_{0L-1} \\ & & & \gamma_{0L} & \frac{2}{3}\Omega_{0L} \end{pmatrix}, \qquad \bar{\mathfrak{B}}_{nn(n>0)} = \begin{pmatrix} \frac{2}{3}\Omega_{n0} & \alpha_{n0} & & & \\ \gamma_{n1} & \frac{2}{3}\Omega_{n1} & \alpha_{n1} & & \\ & \ddots & \ddots & \ddots & \\ & & \gamma_{nL-1} & \frac{2}{3}\Omega_{nL-1} & \alpha_{nL-1} \\ & & & \gamma_{nL} & \frac{2}{3}\Omega_{nL} \end{pmatrix},$$

$$\bar{\mathfrak{D}}_{nn} = -\mathrm{diag}\left(\frac{4+n}{20}, \ldots, \frac{4+n}{20}\right),$$

$$\bar{\mathfrak{D}}_{10} = \frac{1}{20}\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix},$$

$$\bar{\mathfrak{D}}_{nn-1(n>1)} = \mathrm{diag}\left(\frac{n}{20}, \ldots, \frac{n}{20}\right),$$

$$\bar{\mathfrak{B}}_{10} = -\begin{pmatrix} \rho_0 - \frac{1}{30} & & & & \\ \Psi_1 & \rho_1 & & & \\ \phi_2 & \Psi_2 & \rho_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \phi_{L-1} & \Psi_{L-1} & \rho_{L-1} \\ & & & \phi_L & \Psi_L \end{pmatrix}, \qquad \bar{\mathfrak{B}}_{nn-1(n>1)} = -n\begin{pmatrix} \Psi_0 & \rho_0 & & & \\ \phi_1 & \Psi_1 & \rho_1 & & \\ & \ddots & \ddots & \ddots & \\ & & \phi_{L-1} & \Psi_{L-1} & \rho_{L-1} \\ & & & \phi_L & \Psi_L \end{pmatrix},$$

$$\Omega_{nl} = \frac{l(2l+1)(5+2n)}{(4l+3)(4l-1)}, \quad \Psi_l = \frac{4l(2l+1)}{3(4l+3)(4l-1)}.$$