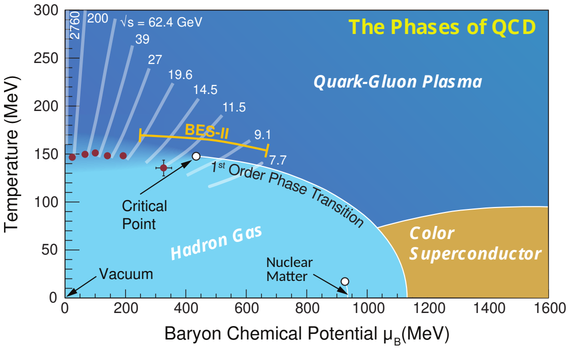
# The critical point, fluctuations, and Hydro+

M. Stephanov

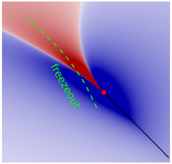
with Y. Yin (MIT/IMP-Lanzhou), 1712.10305;
with X. An, G. Basar and H.-U. Yee, 1902.09517;

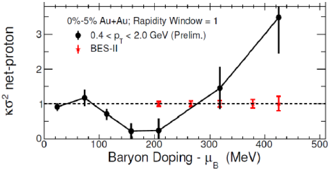
# Critical point: intriguing hints

Where on the QCD phase boundary is the CP?




Equilibrium $\kappa_4$ vs $T$ and $\mu_B$:




"intriguing hint" (2015 LRPNS)

Motivation for phase II of BES at RHIC and BEST topical collaboration.

Theory/experiment gap: predictions assume equilibrium, but

Non-equilibrium physics is essential near the critical point.

Challenge: develop hydrodynamics *with fluctuations* capable of describing *non-equilibrium* effects on critical-point signatures.

Also note:
Fluctuations are the first step to extend hydro to smaller systems.

# Stochastic hydrodynamics

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more

$$\partial_t \breve{\psi} = -\nabla \cdot \Big(\text{Flux}[\breve{\psi}] + \text{Noise}\Big) \qquad \text{(Landau-Lifshitz)}$$

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- Linearized version has been considered and applied to heavy-ion collisions (Kapusta-Muller-MS, Kapusta-Torres-Rincon, . . . )
- Non-linearities + point-like noise ⇒ UV divergences. In numerical simulations – cutoff dependence.

# Deterministic approach

- Variables are one- and two-point functions:
  $\psi = \langle\breve{\psi}\rangle$ and $G = \langle\breve{\psi}\breve{\psi}\rangle - \langle\breve{\psi}\rangle\langle\breve{\psi}\rangle$ – equal-time correlator

$$\partial_t \psi = -\nabla \cdot \mathsf{Flux}[\psi, G]; \qquad \text{(conservation)}$$

$$\partial_t G = \mathsf{L}[G; \psi]. \qquad \text{(relaxation)}$$

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  For arbitrary relativistic flow – by An *et al* (this talk).
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  Earlier, in *nonrelativistic* context, – by Andreev in 1970s.
- Advantage: deterministic equations.

  "Infinite noise" causes UV renormalization of EOS and transport coefficients – can be taken care of *analytically* (*1902.09517*)

# Fluctuation dynamics near CP: Hydro+

*Yin, MS, 1712.10305*

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  - Critical fluctuations ($\xi \to \infty$)
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- Fluctuation dynamics near CP requires two main ingredients:
  - Critical fluctuations ($\xi \to \infty$)
  - Slow relaxation mode with $\tau_{\rm relax} \sim \xi^3$ (leading to $\zeta \to \infty$)
- Both described by the same object: the two-point function of the slowest hydrodynamic mode $\breve{m} = \delta(s/n)$, i.e., $\langle\, \breve{m}(x_1)\, \breve{m}(x_2)\, \rangle$.
- Without this mode, hydrodynamics would break down near CP when $\tau_{\rm expansion} \sim \tau_{\rm relax} \sim \xi^3$.

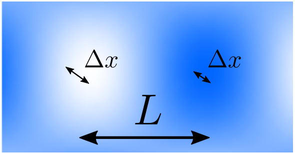
# Additional variables in Hydro+

- At the CP the *slowest* new variable is the 2-pt function $\langle \breve{m}\breve{m} \rangle$ of the slowest hydro variable $\breve{m} = \delta(s/n)$:

$$\phi_{\boldsymbol{Q}}(\boldsymbol{x}) = \int_{\Delta \boldsymbol{x}} \langle \, \breve{m}\,(\boldsymbol{x}_+) \; \breve{m}\,(\boldsymbol{x}_-) \, \rangle \; e^{i\boldsymbol{Q}\cdot\Delta\boldsymbol{x}}$$

  where $\boldsymbol{x} = (\boldsymbol{x}_+ + \boldsymbol{x}_-)/2$ and $\Delta\boldsymbol{x} = \boldsymbol{x}_+ - \boldsymbol{x}_-$.

- Wigner transformed b/c dependence on $\boldsymbol{x}$ ($\sim L$) is much slower than on $\Delta x$. Scale separation similar to kinetic theory.



# Relaxation of fluctuations towards equilibrium

- As usual, equilibration maximizes entropy $S = \sum_i p_i \log(1/p_i)$:

$$s_{(+)}(\epsilon, n, \phi_{\boldsymbol{Q}}) = s(\epsilon, n) + \frac{1}{2}\int_{\boldsymbol{Q}} \left( \log \frac{\phi_{\boldsymbol{Q}}}{\bar{\phi}_{\boldsymbol{Q}}} - \frac{\phi_{\boldsymbol{Q}}}{\bar{\phi}_{\boldsymbol{Q}}} + 1 \right)$$

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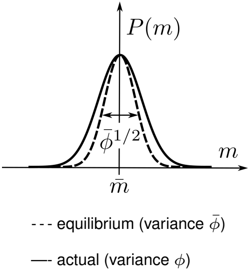
- Entropy = log # of states, which depends on the width of $P(m_{\boldsymbol{Q}})$, i.e., $\phi_{\boldsymbol{Q}}$:
  - Wider distribution – more microstates – more entropy: $\log(\phi/\bar{\phi})^{1/2}$ ;

    vs
  - Penalty for larger deviations from peak entropy (at $\delta m = 0$): $-(1/2)\phi/\bar{\phi}$.




Maximum of $s_{(+)}$ is achieved at $\phi = \bar{\phi}$.

# Hydro+ mode kinetics

- The equation for $\phi_{\boldsymbol{Q}}$ is a relaxation equation:

$$(u\cdot\partial)\phi_{\boldsymbol{Q}} = -\gamma_\pi(\boldsymbol{Q})\pi_{\boldsymbol{Q}}, \quad \pi_{\boldsymbol{Q}} = -\left(\frac{\partial s_{(+)}}{\partial \phi_{\boldsymbol{Q}}}\right)_{\epsilon,n}$$

  $\gamma_\pi(\boldsymbol{Q})$ is known from mode-coupling calculation in 'model H'.

  It is universal (Kawasaki function).

  $\gamma_\pi(\boldsymbol{Q}) \sim 2DQ^2$ for $Q < \xi^{-1}$ and $\sim Q^3$ for $Q > \xi^{-1}$. more

- Characteristic rate: $\Gamma(Q) \sim \gamma_\pi(Q) \sim \xi^{-3}$ at $Q \sim \xi^{-1}$.
- Slowness of this relaxation process is behind the divergence of $\zeta \sim 1/\Gamma \sim \xi^3$ and the breakdown of *ordinary* hydro near CP.

# Towards a general deterministic formalism

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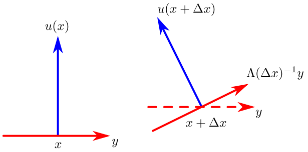
- Important issue in *relativistic* hydro – "equal-time" in the definition of

  $$G(x, y) = \langle\, \phi(x + y/2)\, \phi(x - y/2)\, \rangle.$$

  Addressed by constructing "confluent" derivative.




- Renormalization can be done *analytically*, and resulting renormalized equations are finite (cutoff-independent).

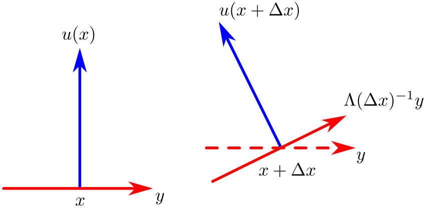
# Equal time

We want evolution equation for equal time correlator $G = \langle \phi(t, \boldsymbol{x}_+)\phi(t, \boldsymbol{x}_-)\rangle$. But what does "equal time" mean?

"Equal time" in $\langle \phi(x_+)\phi(x_-)\rangle$ depends on the choice of frame.

The most natural choice is local $u(x)$ (with $x = (x_+ + x_-)/2$).

Derivatives wrt $x$ at "$y$-fixed" should take this into account:




using $\Lambda(\Delta x)u(x + \Delta x) = u(x)$:

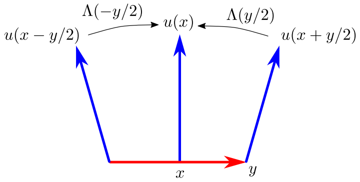
$$\Delta x \cdot \bar{\nabla} G(x, y) \equiv G(x + \Delta x, \Lambda(\Delta x)^{-1} y) - G(x, y)\,.$$

not $G(x + \Delta x, y) - G(x, y)\,.$

# Confluent correlator, derivative and connection

Confluent two-point correlator:

$$\bar{G}(x,y) = \Lambda(y/2)\, G(x,y)\, \Lambda(-y/2)^T$$




and its derivative

$$\bar{\nabla}_\mu \bar{G}_{AB} = \partial_\mu \bar{G}_{AB} - \bar{\omega}^C_{\mu A} \bar{G}_{CB} - \bar{\omega}^C_{\mu B} \bar{G}_{AC} - \mathring{\omega}^b_{\mu a}\, y^a \frac{\partial}{\partial y^b} \bar{G}_{AB}\,.$$

Connection $\bar{\omega}$ makes sure that only the change of $\phi_A$ *relative* to local rest frame $u$ is counted.

Connection $\mathring{\omega}$ corrects for a possible rotation of the local basis triad $e_a$ defining local coordinates $y^a$. The derivative is independent of $e_a$.

We then define the Wigner transform $W_{AB}(x,q)$ of $\bar{G}_{AB}(x,y)$.

# Scales

- Hydro cell size $b$: coarse-grain quantum operators over scale $b \gg \ell_{\rm mic}$ to leave only slow modes for which quantum fluctuatu-ations are negligible compared to thermal, i.e., $\hbar\omega \ll kT$.
  $\ell_{\rm mic} \sim \ell_{\rm mfp},\ c_s/T$.
  $\breve{\psi} = (\,\breve{T}^{i0},\ \breve{J}^0\,)$ are *classical* stochastic variables.
- Hydrodynamic gradients scale $L$: must be $L \gg b$. back
- Size of local equlibrium cell $\ell_{\rm eq} \equiv \ell_*$: diffusion length in evolution time scale, typically $\tau_{\rm ev} \sim L/c_s$

$$\ell_* \sim \sqrt{\gamma\tau_{\rm ev}} \sim \sqrt{\gamma L/c_s}.$$

- $b \ll L$ implies the hierarchy:

$$\ell_{\rm mic} \ll b < \ell_* \ll L \quad \text{or} \quad T/c_s \gg \Lambda > q_* \gg k \quad (\gamma q_*^2 = c_s k)$$

# Matrix equation and diagonalization

After many nontrivial cancellations we find evolution eq.:

$$u \cdot \bar{\nabla} W = -i[\mathbb{L}^{(q)}, W] - \frac{1}{2}\{\bar{\mathbb{L}}, W\} + 2Tw\mathbb{Q}^{(q)} + \mathcal{K} \circ W + \mathcal{K}' \circ q \circ \frac{\partial W}{\partial q}$$

where

expand

Ideal hydro $\rightarrow \mathbb{L}^{(q)} \equiv c_s \begin{pmatrix} 0 & q_\nu \\ q_\mu & 0 \end{pmatrix}, \quad \bar{\mathbb{L}} \equiv c_s \begin{pmatrix} 0 & \bar{\nabla}_{\perp\nu} \\ \bar{\nabla}_{\perp\mu} & 0 \end{pmatrix},$

Noise $\rightarrow \mathbb{Q} \sim \gamma q^2$, and Background $\rightarrow \mathcal{K} \sim \mathcal{K}' \sim \partial_\mu u_\nu$ .

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$$\text{Noise} \to \mathbb{Q} \sim \gamma q^2, \quad \text{and} \quad \text{Background} \to \mathcal{K} \sim \mathcal{K}' \sim \partial_\mu u_\nu \,.$$

The leading term $\mathbb{L}^{(q)}$ is oscillatory: $[\mathbb{L}^{(q)}, W]_{\mathbf{AB}} = (\lambda_{\mathbf{A}} - \lambda_{\mathbf{B}})W_{\mathbf{AB}}$, where $\lambda_{\mathbf{A}} = \pm c_s|q|, 0, 0, 0$, eigenvalues of $\mathbb{L}^{(q)}$ – linear ideal hydro.

Averaging over times shorter than $(c_s|q|)^{-1}$ leaves only 5 modes in $W$: 2 sound-sound $W_{++}$, $W_{--}$ and 2x2 transverse[2] $\widehat{W}_{ij}$. see equations

# Sound-sound correlation and phonon kinetic equation

$$\underbrace{\left[(u+v)\cdot\bar{\nabla} + f\cdot\frac{\partial}{\partial q}\right]W_+}_{\mathcal{L}_+[W_+]} = -\gamma_L q^2(W_+ - \underbrace{Tw}_{W^{(0)}}) + \underbrace{\mathcal{K}''}_{\sim\,\partial_\mu u_\nu,\ a_\mu} W_+$$

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Three nontrivial observations:

- For a phonon $q\cdot u(x) = E(q_\perp)$, where $E = c_s(x)|q_\perp|$:

$$v = c_s \hat{q}_\perp ,$$
$$f_\mu = \underbrace{-E(a_\mu + 2v^\nu \omega_{\nu\mu})}_{\text{inertial + Coriolis}} \underbrace{-q_{\perp\nu}\partial_{\perp\mu}u^\nu}_{\text{“Hubble”}} - \bar{\nabla}_{\perp\mu}E\,.$$

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- Rescaling $N_+ = W_+/(wc_s|q|)$ eliminates $\mathcal{K}''$ terms:

$$\mathcal{L}_+[N_+] = -\gamma_L q^2(N_+ - \underbrace{T/E}_{E\to 0\text{ of eqlbm. BE dist.}})$$

- Contribution of $W_+$ to $T^{\mu\nu}$ matches phonon gas with d.f. $N_+$.

# Renormalization

Expansion of $\langle T^{\mu\nu} \rangle$ contains $\langle \phi(x)\phi(x) \rangle = G(x,0) = \int \frac{d^3q}{(2\pi)^3} W(x,q)$.

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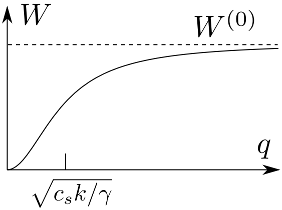


$$W(x,q) \;\sim\; \underbrace{W^{(0)}}_{Tw} \;+\; \underbrace{W^{(1)}}_{\partial u/q^2} \;+\; \widetilde{W}$$

(∼"OPE" or gradient expansion)

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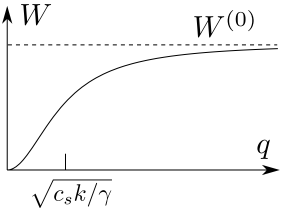


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$$G(x,0) \sim \underbrace{\Lambda^3}_{\text{ideal (EOS)}} + \underbrace{\Lambda\,\partial u}_{\text{visc. terms}} + \underbrace{\widetilde{G}}_{\text{finite “}\partial^{3/2}\text{”}}$$

# Renormalized equations

Expand $T^{\mu\nu}$ in fluctuations and average over noise.

*Local* cutoff-dependent terms absorbed into EOS and visc. coeffs.:

$$T_R^{\mu\nu}(x) = (\epsilon u^\mu u^\nu + p(\epsilon)\Delta^{\mu\nu} + \Pi^{\mu\nu})_R + \frac{1}{w} \underbrace{\left[\left(\dot{c}_s \widetilde{G}_{ee}(x) - c_s^2 \widetilde{G}_\lambda^\lambda(x)\right)\Delta^{\mu\nu} + \widetilde{G}^{\mu\nu}(x)\right]}_{\text{local in } \tilde{G}\text{, but not in } u,\ \epsilon} .$$

And we obtain finite (cutoff independent) system of equations:

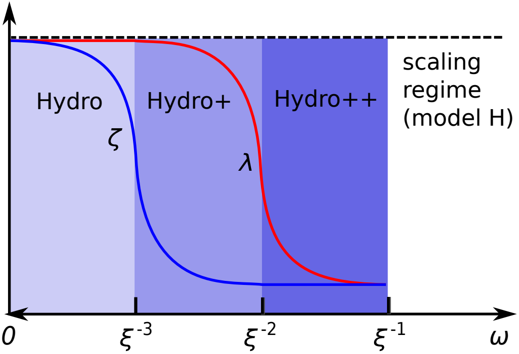
$$\begin{cases} \partial_\mu T_R^{\mu\nu} &= 0\,; \\ u \cdot \bar{\nabla} \widetilde{W} &= \ldots . \end{cases}$$

describing evolution of hydrodynamic variables and their fluctuations.

# In progress

- *Xin An's talk at QM, Wednesday:*

  - Add baryon *charge*.
  - Merge with Hydro+. Unify critical and non-critical fluctuations.




Next-to-slowest modes: density-shear and shear-shear correlator.
Extending Hydro+ closer to the CP (shorter gradients or larger $\xi$).

# Outlook (to-do-list)

- Add higher-order correlators for *non-gaussian* fluctuations.
- Connect *fluctuating* hydro with freezeout kinetics and implement in full hydrodynamic code and event generator.

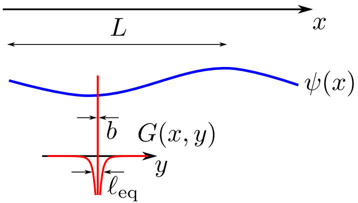
  Compare with experiment.
- First-order transition in fluctuating hydrodynamics?
- Connection to action principle (SK) formulation.

# More

# Separation of scales

$$G(x, y) = \langle\, \phi(x + y/2)\, \phi(x - y/2)\, \rangle$$

depends on $x$ slowly ($L$), but on $y$ – fast ($\ell_{\rm eq} \sim \sqrt{L} \ll L$).



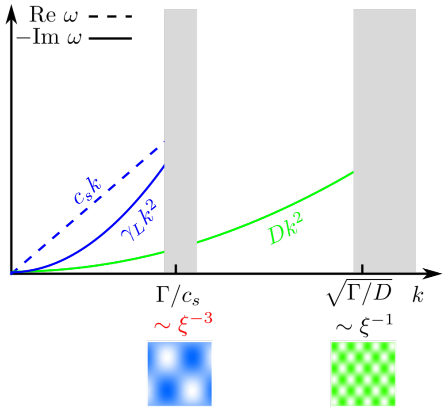

Similar to separation of scales in QFT in kinetic regime. ($q \gg k$)

# Critical fluctuations

- Near CP there is *parametric* separation of relaxation time scales.

  The slowest and thus most out-of-equilibrium mode is charge diffusion at const $p$: $\delta(s/n) \equiv m$.

back



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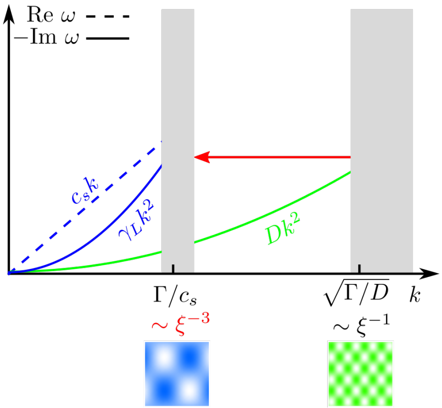
  is of order of that for sound at much smaller $k \sim \xi^{-3}$.



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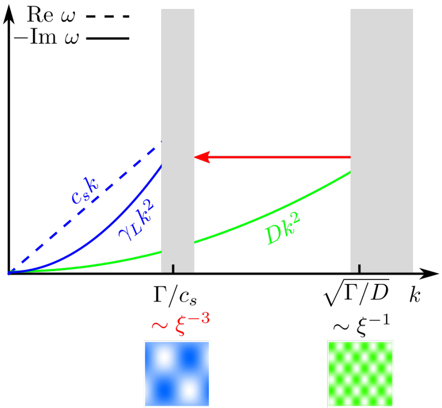

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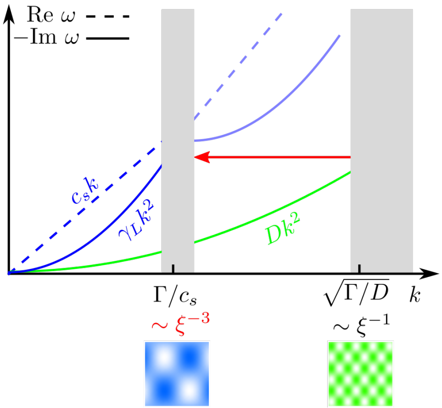
- Thus we need $\langle mm \rangle$ as the independent variable(s) in hydro+ equations.

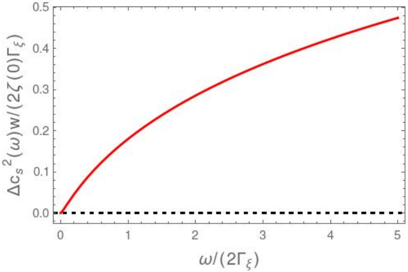


# Hydro+ vs Hydro: real-time bulk response

Hydrodynamics breaks down for processes faster than $\Gamma_\xi \sim \xi^{-3}$ → Hydro+

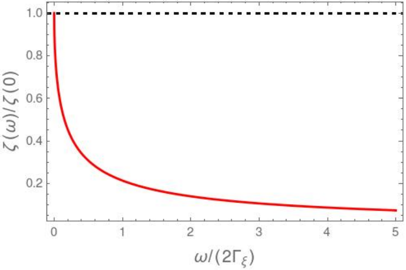
- Stiffness of eos (sound speed) is underestimated in hydro (- - -):

  $c_s \to 0$ at CP, but
  only modes with $\omega \ll \Gamma_\xi$ are critically soft.




- Dissipation during expansion is overestimated in hydro (- - -):

  $\zeta \to \infty$ at CP, but
  only modes with $\omega \ll \Gamma_\xi$ experience large $\zeta$.



# Linearized fluctuation equations

back

$$u \cdot \partial \phi_A = -\big(\mathbb{L} + \mathbb{Q} + \mathbb{K}\big)_{AB}\phi^B - \xi_A \,,$$

where

$$\mathbb{L} \equiv \begin{pmatrix} 0 & c_s\partial_{\perp\nu} \\ c_s\partial_{\perp\mu} & 0 \end{pmatrix}, \quad \mathbb{Q} \equiv \begin{pmatrix} 0 & 0 \\ 0 & -\gamma_\eta \Delta_{\mu\nu}\partial_\perp^2 - (\gamma_\zeta + \frac{1}{3}\gamma_\eta)\partial_{\perp\mu}\partial_{\perp\nu} \end{pmatrix}$$

$$\mathbb{K} \equiv \begin{pmatrix} (1 + c_s^2 + \dot{c}_s)\theta & 2c_s a_\nu \\ \frac{1+c_s^2-\dot{c}_s}{c_s} a_\mu & -u_\mu a_\nu + \partial_{\perp\nu} u_\mu + \Delta_{\mu\nu}\theta \end{pmatrix}, \quad \xi \equiv (0, \Delta_{\mu\kappa}\partial_\lambda \breve{S}^{\lambda\kappa})$$

$$\langle \xi_A(x_+)\xi_B(x_-)\rangle = 2Tw\mathbb{Q}^{(y)}_{AB}\delta^3(y_\perp)\,.$$

$$\begin{aligned} u\cdot\partial G_{AB}(x,y) = &- \Big(\mathbb{L}^{(y)} + \frac{1}{2}\mathbb{L} + \mathbb{Q}^{(y)} + \mathbb{K} + \mathbb{Y}\Big)_{AC} G^C{}_B(x,y) \\ &- \Big(-\mathbb{L}^{(y)} + \frac{1}{2}\mathbb{L} + \mathbb{Q}^{(y)} + \mathbb{K} + \mathbb{Y}\Big)_{BC} G_A{}^C(x,y) \\ &+ 2Tw\mathbb{Q}^{(y)}_{AB}\delta^3(y_\perp), \end{aligned}$$

# Correlation matrix evolution equation

back

$$u\cdot\bar\nabla W(x;q) = -\left[i\mathbb{L}^{(q)}+\mathbb{K}^{(a)},W\right] - \left\{\frac{1}{2}\bar{\mathbb{L}}+\mathbb{Q}^{(q)}+\mathbb{K}^{(s)},W\right\} + \theta W + 2Tw\mathbb{Q}^{(q)} + (\partial_{\perp\lambda}u_\mu)q^\mu\frac{\partial W}{\partial q_\lambda}$$
$$+\frac{1}{2}a_\lambda\left\{\left(1-\frac{\dot c_s}{c_s^2}\right)\mathbb{L}^{(q)},\frac{\partial W}{\partial q_\lambda}\right\} + \frac{\partial}{\partial q_\lambda}\left(\{\Omega_\lambda^{(s)},W\}+[\Omega_\lambda^{(a)},W]-\frac{1}{4}[\mathbb{H}_\lambda,[\mathbb{L}^{(q)},W]]\right),$$

where

$$\mathbb{L}^{(q)}\equiv c_s\begin{pmatrix}0 & q_\nu\\ q_\mu & 0\end{pmatrix},\quad \bar{\mathbb{L}}\equiv c_s\begin{pmatrix}0 & \bar\nabla_{\perp\nu}\\ \bar\nabla_{\perp\mu} & 0\end{pmatrix},\quad \mathbb{Q}^{(q)}\equiv\begin{pmatrix}0 & 0\\ 0 & \gamma_\eta\Delta_{\mu\nu}q^2+\left(\gamma_\zeta+\frac{1}{3}\gamma_\eta\right)q_\mu q_\nu\end{pmatrix},$$

$$\mathbb{K}^{(s)}\equiv\begin{pmatrix}(1+c_s^2+\dot c_s)\,\theta & \frac{1}{2c_s}(1+2c_s^2)\,a_\nu\\ \frac{1}{2c_s}(1+2c_s^2)\,a_\mu & \Delta_{\mu\nu}\theta+\theta_{\mu\nu}\end{pmatrix},\quad \mathbb{K}^{(a)}\equiv\begin{pmatrix}0 & -\frac{1-c_s^2-\dot c_s}{2c_s}a_\nu\\ \frac{1-c_s^2-\dot c_s}{2c_s}a_\mu & -\omega_{\mu\nu}\end{pmatrix},$$

$$\Omega_\lambda^{(s)}\equiv\frac{c_s^2}{2}\begin{pmatrix}2\omega_{\kappa\lambda}q^\kappa & 0\\ 0 & \omega_{\mu\lambda}q_\nu+\omega_{\nu\lambda}q_\mu\end{pmatrix},\quad \Omega_\lambda^{(a)}\equiv\frac{c_s^2}{2}\begin{pmatrix}0 & 0\\ 0 & \omega_{\mu\lambda}q_\nu-\omega_{\nu\lambda}q_\mu\end{pmatrix},$$

$$\mathbb{H}_\lambda\equiv c_s\begin{pmatrix}0 & \partial_\nu u_\lambda\\ \partial_\mu u_\lambda & 0\end{pmatrix},$$

$$\theta^{\mu\nu}=\frac{1}{2}\left(\partial_\perp^\mu u^\nu+\partial_\perp^\nu u^\mu\right),\quad \theta=\theta_\mu^\mu,\quad \omega_{\mu\nu}=\frac{1}{2}(\partial_{\perp\mu}u_\nu-\partial_{\perp\nu}u_\mu).$$

# Wigner function equations

go back

Sound-sound

$$(u \pm c_s \hat{q}) \cdot \bar{\nabla} W_\pm - \left( \pm \left( c_s - \frac{\dot{c}_s}{c_s} \right) |q| a_\mu + (\partial_{\perp\mu} u_\nu) q^\nu + 2c_s^2 q^\lambda \omega_{\lambda\mu} \right) \frac{\partial W_\pm}{\partial q_\mu}$$
$$= -\gamma_L q^2 (W_\pm - Tw) - \left( (1 + c_s^2 + \dot{c}_s)\theta + \theta_{\mu\nu} \hat{q}^\mu \hat{q}^\nu \pm \frac{1 + 2c_s^2}{c_s} \hat{q} \cdot a \right) W_\pm \, ,$$

# Wigner function equations

Sound-sound

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Shear-shear

$$u \cdot \bar{\nabla} \widehat{W} = -2q^2 \gamma_\eta (\widehat{W} - Tw\widehat{\mathbb{1}}) + (\partial_{\perp\mu} u_\nu) q^\nu \nabla^\mu_{(q)} \widehat{W} - \left\{ \widehat{K}, \widehat{W} \right\} + \left[ \widehat{\Omega}, \widehat{W} \right] \,,$$

where

$$\widehat{K}^{ij} \equiv \frac{1}{2}\theta \, \delta^{ij} + \theta^{\mu\nu} t^{(i)}_\mu t^{(j)}_\nu, \quad \text{and} \quad \widehat{\Omega}^{ij} \equiv \omega^{\mu\nu} t^{(i)}_\mu t^{(j)}_\nu, \quad i = 1, 2;$$

go back

# Large $q$ behavior of $W$

The part which does not lead to UV divergences:

$$\widetilde{W} = W - W^{(0)} - W^{(1)}$$

The equilibrium part (the divergent integral renormalizes EOS):

$$W_{\pm}^{(0)} = Tw \quad \text{and} \quad W_{T_i,T_j}^{(0)} = Tw\delta_{ij}.$$

The first background gradient correction
(integral renormalizes viscosities):

$$\begin{aligned} W_{\pm}^{(1)}(x,q) &= \frac{Tw}{\gamma_L q^2}\left((c_s^2 - \dot{c}_s)\theta - \theta_{\mu\nu}\hat{q}^{\mu}\hat{q}^{\nu}\right) , \\ W_{T_iT_j}^{(1)}(x,q) &= \frac{Tw}{\gamma_\eta q^2}\left(c_s^2\theta\,\delta^{ij} - \theta^{\mu\nu}t_{\mu}^{(i)}t_{\nu}^{(j)}\right) . \end{aligned}$$

back

# Universality and mapping of QCD to Ising model

- The EOS is an essential input for hydro.

  Near CP universality means

  $$P_{\mathrm{QCD}}(\mu, T) = -G_{\mathrm{Ising}}(h, r) + \text{less singular terms}$$

  $G_{\mathrm{Ising}}(h, r)$ is universal and known,

  but the mapping given by $h = h(\mu, T)$ and $r = r(\mu, T)$ is not.

# Universality and mapping of QCD to Ising model

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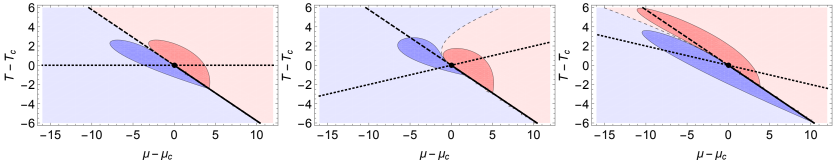
  but the mapping given by $h = h(\mu, T)$ and $r = r(\mu, T)$ is not.

- While $h = 0$ is the transition line, what is $r = 0$?

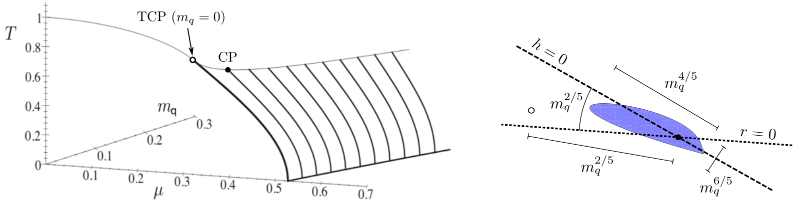
  Slope of $r = 0 \Leftrightarrow$ asymmetry of EOS around transition line:




The skewness, or $\chi_3$, can be 0, + or - depending on $r = 0$ slope.

# Universality of mapping for small $m_q$

- In the limit of $m_q \to 0$ the critical point is close to a tricritical point.




- The $(\mu, T)/(h, r)$ mapping becomes singular in a *universal* way: the slope difference vanishes as $\sim m_q^{2/5}$. *Pradeep, MS, 1905.13247*

  Consequences:

  - The $r = 0$ axis is almost horizontal. Not $\perp$ to $h = 0$.
  - $r = 0$ slope is possibly negative (it is in RMM). Then skewness is negative on the crossover line ($h = 0$) and below, at freezeout.