Chiral kinetic theory in curved spacetime

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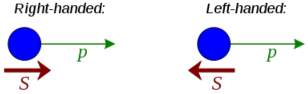
Chiral magnetic effect and chiral vortical effect

- Chirality is the same as helicity for massless particles: the sign of the projection of the spin vector onto the momentum vector (picture from wikipedia)



- Chiral magnetic effect(CME)(Kharzeev, Mclerran, Warringa, Fukushima 2008; Son, Zhitnitsky 2004)

$$\begin{aligned} \mathbf{J} &= \sigma \mathbf{B}, \quad \sigma = \frac{1}{2\pi^2}\mu_5, \\ \mathbf{J}_5 &= \sigma_5 \mathbf{B}, \quad \sigma_5 = \frac{1}{2\pi^2}\mu, \end{aligned} \tag{1}$$

with **B** the magnetic field.

- Chiral vortical effect(CVE)(Erdmenger etal 2008; Barnerjee etal 2008, Son, Surowka 2009; Landsteiner etal 2011)

$$\begin{aligned} \mathbf{J} &= \xi \overrightarrow{\omega}, \quad \xi = \frac{1}{\pi^2}\mu\mu_5, \\ \mathbf{J}_5 &= \xi_5 \overrightarrow{\omega}, \quad \xi_5 = \frac{(\mu^2 + \mu_5^2)}{2\pi^2} + \frac{T^2}{6}, \end{aligned} \tag{2}$$

with $\overrightarrow{\omega}$ the fluid vorticity.

Rotating fluid vs. rotating frame

- Lorentz force $\Longleftrightarrow$ Coriolis force in rotating frame

In magnetic field, Lorentz force:
$$\boldsymbol{F} = e(\dot{\boldsymbol{x}} \times \boldsymbol{B})$$

In rotating frame, Coriolis force:
$$\boldsymbol{F} = 2\boldsymbol{\varepsilon}(\dot{\boldsymbol{x}} \times \boldsymbol{\omega}) + \boldsymbol{O}(\boldsymbol{\omega}^2)$$

- Correspondence: $\mathbf{B} \Longleftrightarrow 2\epsilon\overrightarrow{\omega}$ (Stephanov, Yin 2012)

$$\mathbf{J}_{CME} = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\mathbf{B}(\mathbf{b}\cdot\hat{\mathbf{p}})f \quad \Leftrightarrow \quad \mathbf{J}_{CVE} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} 2\epsilon\overrightarrow{\omega}(\mathbf{b}\cdot\hat{\mathbf{p}})f, \quad (3)$$

where $\mathbf{b} = \frac{\hat{\mathbf{p}}}{2|\mathbf{p}|^2}$ is the Berry curvature. This ($+$ antiparticle) indeed gives expected result.

A kinetic theory in general spacetime would be helpful to study rotating frame and CVE.

Wigner operator and Horizontal lift

- Wigner operator

$$\hat{W}_{\alpha\beta}(x,p) \equiv \int \frac{\sqrt{-g}d^4y}{(2\pi)^4} e^{-ip\cdot y/\hbar} \left[\bar{\psi}(x) e^{1/2y\cdot\overleftarrow{\nabla}}\right]_{\beta} \left[e^{-1/2y\cdot\nabla}\psi(x)\right]_{\alpha}, \qquad (4)$$

where the derivative $\overleftarrow{\nabla}_\mu(\nabla_\mu)$ acting to the left(right). We emphasize that x in equation (4) is the coordinate of point(P) in curved spacetime, and y is vector in the tangent space of point P, and p is vector in cotangent space of P.

- Horizontal lifted covariant derivatives (Winter. 1985; Calzetta, Habib, Hu. 1988; Fonarev. 1994)

$$\nabla_\mu \equiv D_\mu - \Gamma^\lambda_{\mu\nu} y^\nu \frac{\partial}{\partial y^\lambda} + \Gamma^\lambda_{\mu\nu} p_\lambda \frac{\partial}{\partial p_\nu} \underbrace{+\Gamma_\mu + \frac{i}{\hbar} A_\mu}_{\text{connection for spinor}}, \qquad (5)$$

$$\overleftarrow{\nabla}_\mu \equiv \overleftarrow{D}_\mu - \frac{\overleftarrow{\partial}}{\partial y^\lambda} \Gamma^\lambda_{\mu\nu} y^\nu + \frac{\overleftarrow{\partial}}{\partial p_\nu} \Gamma^\lambda_{\mu\nu} p_\lambda \overbrace{-\Gamma_\mu - \frac{i}{\hbar} A_\mu}, \qquad (6)$$

where D_μ is the usual covariant derivative operator, A_μ is gauge field, $\Gamma_\mu \equiv -\frac{i}{4}\omega^{ab}_\mu \sigma_{ab}$ is spin connection with $\sigma_{ab} = \frac{i}{2}[\gamma_a, \gamma_b]$ and ω^{ab}_μ the vierbein connection.

- Vierbein: $e^a = e^a_\mu \partial^\mu$.

Dynamic equation for Wigner function

- Dirac equation

$$\gamma^\mu \nabla_\mu \psi = 0 = \bar{\psi} \overleftarrow{\nabla}_\mu \gamma^\mu . \tag{7}$$

- Dynamic equation up to $O(\hbar^2)$

$$\begin{aligned} \gamma^\mu \left(\Pi_\mu + \frac{i\hbar}{2} \Delta_\mu \right) \hat{W} &= \frac{i\hbar^2}{32} \gamma^\mu R_{\mu\alpha\rho\sigma} \left[\partial_p^\alpha \hat{W}, \sigma^{\mu\nu} \right] \\ &\quad - \frac{\hbar^3}{8 \times 4!} (\nabla_\beta R_{\mu\alpha\rho\sigma}) \gamma^\mu \left[\partial_p^\alpha \partial_p^\beta \hat{W}, \sigma^{\rho\sigma} \right] \end{aligned} \tag{8}$$

with

$$\begin{aligned} \Pi_\mu &= p_\mu - \frac{\hbar^2}{12} (\nabla_\rho F_{\mu\nu}) \partial_p^\nu \partial_p^\rho + \frac{\hbar^2}{24} R^\rho{}_{\sigma\mu\nu} \partial_p^\sigma \partial_p^\nu p_\rho + \frac{\hbar^2}{4} R_{\mu\nu} \partial_p^\nu , \\ \Delta_\mu &= \nabla_\mu - F_{\mu\lambda} \partial_p^\lambda - \frac{\hbar^2}{12} (\nabla_\rho R_{\mu\nu}) \partial_p^\rho \partial_p^\nu - \frac{\hbar^2}{24} (\nabla_\lambda R^\rho{}_{\sigma\mu\nu}) \partial_p^\nu \partial_p^\sigma \partial_p^\lambda p_\rho \\ &\quad + \frac{\hbar^2}{8} R^\rho{}_{\sigma\mu\nu} \partial_p^\nu \partial_p^\sigma \nabla_\rho + \frac{\hbar^2}{24} (\nabla_\alpha \nabla_\beta F_{\mu\nu} + 2 R^\rho{}_{\alpha\mu\nu} F_{\beta\rho}) \partial_p^\nu \partial_p^\alpha \partial_p^\beta , \end{aligned} \tag{9}$$

where $R^\mu{}_{\nu\rho\sigma}$ is Riemann curvature and $R_{\mu\nu}$ is Ricci tensor.

Chiral kinetic theory up to $O(\hbar)$

- kinetic equation

$$0 = \delta(p^2 - \hbar F_{\alpha\beta}\Sigma^{\alpha\beta})\left[p_\mu \triangle^\mu f_r - \frac{\hbar}{p\cdot n}\widetilde{F}_{\mu\nu} n^\nu \triangle^\mu f_r + \underbrace{\hbar \triangle^\mu (\Sigma_{\mu\nu}\triangle^\nu f_r)}_{spin-curvature\ coupling}\right] \quad (10)$$

  - $\triangle_\mu = \nabla_\mu - F_{\mu\nu}\partial_p^\nu$;
  - n^μ is a frame choosing vector that satisfies $n^2 = 1$;
  - $\Sigma_{\mu\nu} \equiv \frac{1}{2p\cdot n}\epsilon_{\mu\nu\rho\sigma}p^\rho n^\sigma$ is the spin tensor.

- Current

$$J^\mu = 4\pi \int \frac{d^4p}{\sqrt{-g}(2\pi)^4}\delta(p^2 - \frac{\hbar p^\sigma B_\sigma}{p\cdot n})\left[p_\mu f_r - \hbar\Sigma_{\mu\nu}\left(\frac{E^\nu}{p\cdot n} - \nabla^\nu\right)f_r\right], \quad (11)$$

where $E_\mu = n^\nu F_{\mu\nu}$ and $B_\mu = n^\nu \widetilde{F}_{\mu\nu}$.

Equilibrium state

We consider the equilibrium distribution $f_r(g)$, with g a linear combination of collisional conserved quantities.

- Conserved quantities are the particle number, the momentum and the angular momentum

$$g = \alpha_R(x) + \beta^\mu(x) p_\mu + \hbar \Sigma_{\mu\nu}\gamma^{\mu\nu} \tag{12}$$

- Equilibrium conditions

$$\begin{aligned} \nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu &= \phi(x) g_{\mu\nu}, & (13) \\ \gamma_{\mu\nu} - \frac{1}{2}\nabla_\mu \beta_\nu &= 0, & (14) \\ \nabla_\mu \alpha_R &= F_{\mu\nu}\beta^\nu. & (15) \end{aligned}$$

  - One can identify $\alpha_R = \beta\mu_R$ and $\beta^\mu = \beta u^\mu$, where u^μ is the four velocity of the volume element.
  - For $\phi = constant$, we have $\beta^\mu = \beta^\mu_{con} + \underbrace{\omega^{\mu\nu} x_\nu}_{rotation} + \underbrace{\phi x^\mu}_{expansion}$.

Abelian gauge field in Minkowski space

- kinetic equation with $n^\mu = (1,0,0,0)$

$$
\begin{aligned}
0 \quad = \quad & \left\{\left(1+\frac{\hbar(\mathbf{B}\cdot\mathbf{p})}{2|\mathbf{p}|^3}\right)\partial_t + \left(\mathbf{v}+\frac{\hbar\mathbf{B}}{2|\mathbf{p}|^2}+\frac{\hbar}{2|\mathbf{p}|^3}[(\mathbf{E}-\nabla\epsilon_p)\times\mathbf{p}]\right)\cdot\nabla \right. \\
& \left. +(\mathbf{E}-\nabla\epsilon_p)\cdot\nabla_{\mathbf{p}} + \mathbf{v}\times\mathbf{B}\cdot\nabla_{\mathbf{p}} + \frac{\hbar}{2|\mathbf{p}|^3}[(\mathbf{E}-\nabla\epsilon_p)\cdot\mathbf{B}]\mathbf{p}\cdot\nabla_{\mathbf{p}}\right\} f_r \qquad (16)
\end{aligned}
$$

where $\epsilon_p \equiv p_0 = |\mathbf{p}| - \frac{\hbar\mathbf{B}\cdot\mathbf{p}}{2|\mathbf{p}|^2}$ and $\mathbf{v} \equiv \frac{\partial\epsilon_p}{\partial\mathbf{p}}$. (Son, Yamamoto. 2013; Hidaka, Pu, Yang. 2017; Huang, Shi, Jiang, et al. 2018.)

- current

$$
\begin{aligned}
J^0 \quad &= \quad \int d^3\mathbf{p}\left(1+\frac{\hbar(\mathbf{B}\cdot\mathbf{p})}{2|\mathbf{p}|^3}\right)f_r, \\
\mathbf{J} \quad &= \quad \int d^3\mathbf{p}\left(\mathbf{v}+\frac{\hbar\mathbf{B}}{2|\mathbf{p}|^2}+\frac{\hbar}{2|\mathbf{p}|^3}\mathbf{E}\times\mathbf{p}-\frac{\hbar}{2|\mathbf{p}|^3}\epsilon_{\mathbf{p}}\mathbf{p}\times\nabla\right)f_r \qquad (17)
\end{aligned}
$$

Rotating fluid in Minkowski space

- $\epsilon_p \equiv p_0 = |\mathbf{p}|$ and $\mathbf{v} \equiv \frac{\partial \epsilon_p}{\partial \mathbf{p}}$.
- kinetic equation with $n^\mu = u^\mu = (1, \boldsymbol{\Omega} \times \mathbf{x})$ the velocity vector of fluid

$$0 = \sqrt{G}\left[\partial_t + \left(\mathbf{v} + \frac{\hbar \boldsymbol{\Omega}}{2|\mathbf{p}|} - \frac{\hbar \mathbf{p} \cdot \boldsymbol{\Omega}}{2|\mathbf{p}|^3}\mathbf{p}\right) \cdot \nabla\right] f_r, \quad (18)$$

with $\boldsymbol{\Omega}$ the angular velocity of fluid and $\sqrt{G} \equiv \left(1 + \hbar \frac{\mathbf{p}\cdot\boldsymbol{\Omega}}{|\mathbf{p}|^2}\right)$.

- Single particle EOM

$$\sqrt{G}\dot{\mathbf{x}} = \frac{\partial \widetilde{\epsilon_{\mathbf{p}}}}{\partial \mathbf{p}} + \frac{\hbar \boldsymbol{\Omega}}{|\mathbf{p}|}, \qquad \sqrt{G}\dot{\mathbf{p}} = 0, \quad (19)$$

where $\widetilde{\epsilon_{\mathbf{p}}} \equiv \epsilon_p - \frac{\hbar}{2}\hat{\mathbf{p}} \cdot \boldsymbol{\Omega} = |\mathbf{p}| - \frac{\hbar}{2}\hat{\mathbf{p}} \cdot \boldsymbol{\Omega}$.

- Equilibrium distribution function

$$f_r^{eq} = f_r(\mu + |\mathbf{p}| - \mathbf{p} \cdot (\boldsymbol{\Omega} \times \mathbf{x}) - \frac{\hbar}{2|\mathbf{p}|}\mathbf{p} \cdot \boldsymbol{\Omega}) \quad (20)$$

- The CVE current (Chen, Son, Stephanov, et al. 2014)

$$\mathbf{J} = \int d^3\mathbf{p}\left\{\mathbf{v} - \hbar \frac{\mathbf{p}}{2|\mathbf{p}|^2} \times \nabla\right\} f_r. \quad (21)$$

Rotating frame with angular velocity $\boldsymbol{\Omega}$

- Metric

$$g_{00} = 1 - (\boldsymbol{\Omega} \times \mathbf{x})^2, \quad g_{0i} = g_{i0} = -(\boldsymbol{\Omega} \times \mathbf{x})^i, \quad g_{ij} = -\delta_{ij}. \tag{22}$$

$$g^{00} = 1, \quad g^{0i} = g^{i0} = -(\boldsymbol{\Omega} \times \mathbf{x})^i, \quad g^{ij} = -\delta^{ij} + (\boldsymbol{\Omega} \times \mathbf{x})^i (\boldsymbol{\Omega} \times \mathbf{x})^j. \tag{23}$$

- Nonzero connections

$$\Gamma^i{}_{00} = [\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x})]^i, \qquad \Gamma^i{}_{0j} = \Gamma^i{}_{j0} = \epsilon^{ilj} \Omega^l. \tag{24}$$

- Riemann curvature is identically zero in rotating frame.

Setup

- Momentum

$$\mathbf{p} \equiv -(p_1, p_2, p_3) \tag{25}$$

- Energy

$$\epsilon_{\mathbf{p}} \equiv p_0 = |\mathbf{p}| - (\boldsymbol{\Omega} \times \mathbf{x}) \cdot \mathbf{p}, \qquad p^0 = g^{0\nu} p_\nu = g^{00} p_0 + g^{0i} p_i = |\mathbf{p}|. \tag{26}$$

- Effective velocity

$$\mathbf{v} \equiv \frac{\partial \epsilon_{\mathbf{p}}}{\partial \mathbf{p}} = \frac{\mathbf{p}}{|\mathbf{p}|} - (\boldsymbol{\Omega} \times \mathbf{x}) \tag{27}$$

Co-rotating observer and Inertial fluid

- Kinetic equation with $n^\mu = u^\mu = (1, -(\boldsymbol{\Omega} \times \mathbf{x}))^\mu$

$$0 = \left\{ \partial_t + \mathbf{v} \cdot \nabla + \mathbf{p} \times \boldsymbol{\Omega} \frac{\partial}{\partial \mathbf{p}} \right\} f_r. \tag{28}$$

- EOM

$$\begin{aligned} \dot{\mathbf{x}} &= \hat{\mathbf{p}} - (\boldsymbol{\Omega} \times \mathbf{x}) = \frac{\partial \epsilon_{\mathbf{p}}}{\partial \mathbf{p}} \\ \dot{\mathbf{p}} &= \mathbf{p} \times \boldsymbol{\Omega} = -\frac{\partial \epsilon_{\mathbf{p}}}{\partial \mathbf{x}} \end{aligned} \tag{29}$$

- Current

$$J^0 = \int d^3\mathbf{p} f_r \tag{30}$$

$$\mathbf{J} = \int d^3\mathbf{p} \left(\mathbf{v} f_r - \frac{\hbar \mathbf{p}}{2|\mathbf{p}|^2} \times \nabla f_r \right) \tag{31}$$

- Equilibrium distribution function

$$f_r = f_r(\mu + |\mathbf{p}|) \tag{32}$$

we find there is no CVE.

Co-rotating observer and Co-rotating fluid

- kinetic equation with $n^\mu = u^\mu = (1,0,0,0)$

$$0 = \left\{\sqrt{G}\partial_t + \sqrt{G}\left[\mathbf{v} + \hbar\left(\frac{\mathbf{\Omega}}{2|\mathbf{p}|} - \frac{(\mathbf{p}\cdot\mathbf{\Omega})\mathbf{p}}{2|\mathbf{p}|^3}\right)\right]\cdot\nabla + \mathbf{p}\times\mathbf{\Omega}\cdot\frac{\partial}{\partial\mathbf{p}}\right\}f_r. \quad (33)$$

with $\sqrt{G} \equiv (1 + \hbar\frac{\mathbf{p}\cdot\mathbf{\Omega}}{|\mathbf{p}|^2})$.

- Single particle EOM

$$\begin{aligned}\sqrt{G}\dot{\mathbf{x}} &= \frac{\partial\widetilde{\epsilon_{\mathbf{p}}}}{\partial\mathbf{p}} + \frac{\hbar\mathbf{\Omega}}{|\mathbf{p}|},\\ \sqrt{G}\dot{\mathbf{p}} &= \mathbf{p}\times\mathbf{\Omega}, \end{aligned} \quad (34)$$

with $\widetilde{\epsilon_{\mathbf{p}}} \equiv \epsilon_{\mathbf{p}} - \frac{\hbar}{2}\hat{\mathbf{p}}\cdot\mathbf{\Omega} = |\mathbf{p}| - \mathbf{p}\cdot(\mathbf{\Omega}\times\mathbf{x}) - \frac{\hbar}{2}\hat{\mathbf{p}}\cdot\mathbf{\Omega}$

- Current

$$\mathbf{J} = \int d^3\mathbf{p}\left(\mathbf{v}f_r - \frac{\hbar\mathbf{p}}{2|\mathbf{p}|^2}\times\nabla f_r\right) \quad (35)$$

- Equilibrium state

$$f_r^{eq} = f_r(\mu + |\mathbf{p}| - \mathbf{p}\cdot(\mathbf{\Omega}\times\mathbf{x}) - \frac{\hbar}{2}\hat{\mathbf{p}}\cdot\mathbf{\Omega}), \quad (36)$$

it induces CVE.

Co-rotating observer and Co-rotating fluid

One can find the Coriolis force

$$|\mathbf{p}|\ddot{\mathbf{x}} = 2\mathbf{p} \times \mathbf{\Omega}. \tag{37}$$

Redefining the three components momentum $\mathbf{k} = \mathbf{p} - |\mathbf{p}|\mathbf{\Omega} \times \mathbf{x}$

- Kinetic equation with $\sqrt{G} = \left(1 + \hbar \frac{\mathbf{k}\cdot\mathbf{\Omega}}{|\mathbf{k}|^2}\right)$.

$$0 = \left\{\sqrt{G}\partial_t + \sqrt{G}\left[\hat{\mathbf{k}} + \hbar\left(\frac{\mathbf{\Omega}}{2|\mathbf{k}|} - \frac{(\mathbf{k}\cdot\mathbf{\Omega})\mathbf{k}}{2|\mathbf{k}|^3}\right)\right] \cdot \nabla + 2\mathbf{k} \times \mathbf{\Omega} \cdot \frac{\partial}{\partial \mathbf{k}}\right\} f_r \tag{38}$$

- Single particle EOM

$$\begin{aligned} \sqrt{G}\dot{\mathbf{x}} &= \frac{\partial \widetilde{\epsilon}_k}{\partial \mathbf{k}} + \frac{\hbar\mathbf{\Omega}}{|\mathbf{k}|}, \\ \sqrt{G}\dot{\mathbf{k}} &= 2\mathbf{k} \times \mathbf{\Omega}, \end{aligned} \tag{39}$$

with $\widetilde{\epsilon}_{\mathbf{k}} = |\mathbf{k}| - \frac{\hbar\mathbf{\Omega}\cdot\mathbf{k}}{2|\mathbf{k}|}$

Compare with magnetic field case

$$\begin{aligned} \sqrt{G_{\mathbf{B}}}\dot{\mathbf{x}} &= \frac{\partial \epsilon_k}{\partial \mathbf{k}} + \frac{\hbar\mathbf{B}}{2|\mathbf{k}|^2} \\ \sqrt{G_{\mathbf{B}}}\dot{\mathbf{k}} &= \hat{\mathbf{k}} \times \mathbf{B} \end{aligned} \tag{40}$$

with $\sqrt{G_{\mathbf{B}}} = \left(1 + \frac{\hbar(\mathbf{B}\cdot\mathbf{k})}{2|\mathbf{k}|^3}\right)$ and $\epsilon_k \equiv |\mathbf{k}| - \frac{\hbar\mathbf{B}\cdot\mathbf{k}}{2|\mathbf{k}|^2}$

Correspondence: $\mathbf{B} \Longleftrightarrow 2|\mathbf{k}|\mathbf{\Omega}$.

Summary and outlook

Summary

- We have derived the covariant chiral kinetic theory up to $O(\hbar)$ in curved spacetime.
- We found that the CVE is due to rotating of the fluid in inertial frame, and a rotating frame itself does not generate the CVE.
- We have derived the correspondence between the magnetic field and the vorticity.

Outlook

- Quantum correction to the collision term.
- Expanding fluid.
- Spin dynamics.

Thank you!